 Goals of this lecture:

- to show how describing functions may be used in stability analysis of closed loop systems

Refs:

SASTRY §4.1
Using Describing Functions in Stability Studies

**WARNING:** block diagram manipulations for nonlinear systems.

\[ a \xrightarrow{\text{asint}} f_1 \xrightarrow{+} f_2 \xrightarrow{+} \ldots \xrightarrow{+} \exists \]

- first harmonic terms from each \( f_i \) add to give first harmonic from \( f \).

Nonlinear systems \( f_i \) connected in parallel

\[ \Rightarrow N_f(a) = N_{f_1}(a) + N_{f_2}(a) + \ldots + N_{f_n}(a) \]

**Application:** if we can decompose a given \( f \) as

\[ f(x) = \exists f_i(x) \]

for \( f_i \) having known \( N_{f_i} \), then

\[ N_f(a) = \exists N_{f_i}(a) \text{ (easy to find)} \]

**However:**

\[ \xrightarrow{\text{f1}} \xrightarrow{\text{f2}} \]

**WARNING:** \( N_f(a) \neq N_{f_1}(a) \cdot N_{f_2}(a) \)

in general

and similarly for several functions \( f_i \) in Series.
Also: for linear systems.

But for nonlinear systems

because, for nonlinear \( n \), superposition is not valid.

For nonlinear systems, manipulations are OK iff inputs to nonlinearities remain unaltered.

\[ \text{eg.} \]
Many nonlinear systems can be written as:

\[ 0 \xrightarrow{+e} n \xrightarrow{v} g(s) \xrightarrow{} y \]

where \( n \) is a SVSS or DVSS function.

Concentrate on error \( e \): since if we know what happens to \( e \), we can determine what happens elsewhere in the system, easily.

- will it oscillate?
- what amplitude?
- what \( \omega \)?

Assume \[ e = \text{sustained sinusoidal oscillation} \]

\[ \Rightarrow e(t) = a \sin(\omega t + \theta) \] for some \( a, \omega, \theta \) to be estimated

Then \[ v(t) = n(e(t)) \]

\[ = n(a \sin(\omega t + \theta)) \]

\[ = a |N(a)| \sin(\omega t + \theta + \varphi(a)) + \text{higher order terms} \]

where \( N(a) \) is the transfer function between the input to \( n \) and the 1st harmonic in the output of \( n \), and \( \varphi(a) \) is the phase angle associated with \( N(a) \).
\[ e(t) = a \sin (wt + \theta) \]
\[ v(t) = a |N(a)| \sin (wt + \theta + \varphi(a)) + \text{higher harmonics.} \]

Assume: \( g(s) \) attenuates higher harmonics, in that \( g \) looks like: \(|g|\)

Then \( y(t) = \text{result of } g \text{ operating on } 1\text{st harmonic in } v \)
\[ = |g(jw)| a |N(a)| \sin (wt + \theta + \varphi(a) + \gamma(w)) \]

So \( e(t) = -y(t) \)
\[ \Rightarrow a \sin (wt + \theta) = -|g(jw)| a |N(a)| \sin (wt + \theta + \varphi(a) + \gamma(w)) \]

\[ \text{assume this is actually } = \]
\[ a \sin (wt + \theta) = -|g(jw)| a |N(a)| \sin (wt + \theta + \varphi(a) + \gamma(w)) \]

In phaser form:
\[ ae^{i(wt+\theta)} = -|g(jw)| a |N(a)| e^{i(wt + \theta + \varphi(a) + \gamma(w))} \]
\[ = -|g(jw)| e^{\gamma(w)}. |N(a)| e^{i\varphi(a)} e^{i(wt+\theta)} \]
\[ 1 = \frac{\left| g(jw) \right| e^{j\gamma(w)} \cdot |N(a)| e^{j\phi(a)}}{g(jw) N(a)} \]

Harmonic balance equation

**Example:**

\[ g(jw) = -\frac{1}{N(a)} \]

Relationship between \( w \) and \( a \)

Assumptions made:

- \( e(t) = a \sin (wt + \theta) \)
- \( g \) attenuates higher frequencies
- Describing function is exact.

Assume:

\[ g(s) = \frac{4\theta}{s(s+2)(s+8)} \]

Suggests: if intersection takes place, then oscillations may occur in \( e \) with the \( w, a \) associated with the intersection point.
But if we had:

\[ g(jw) = \frac{-1}{N(a)} \] not true for any \( w, a > 0 \)

which suggests no oscillation (because if \( e(t) \equiv a \sin(wt + \theta) \) and assumptions all valid, then \( g(jw) = \frac{-1}{N(a)} \) must be true for some \( w, a \).

**Note:** Predictions not foolproof in that, owing to the approximations involved in the analysis:

- Predicted oscillations might not happen.
- Predicted no-oscillations might be false.

Nonetheless, this is often a useful tool.

Q: Can we predict whether oscillations will
- decay
- be sustained
- explode

A: Yes... using an extension of Nyquist.
Nyquist:

\[
\begin{array}{c}
\leq \\
\rightarrow \\
K \\
\rightarrow \\
g(s)
\end{array}
\]

Then: closed loop poles are in open left half plane iff complete \( g(jw) \) - locus encircles \( \left[ -\frac{1}{K} + j0 \right] \) \( p \) times (anticlockwise)

\# OL poles in open RHP

\[ p = 0 \]

\(-\frac{1}{K} \) here: at least one closed loop pole in closed rhp

Actually, true for all \( k \in C \):

\[ p = 0 \]

\(-\frac{1}{K} \) here: all \( \frac{1}{K} \) poles in open rhp

\(-\frac{1}{K} \) here: at least one closed loop pole in closed rhp

\( p = 0 \)

\(-\frac{1}{K} \) here: at least one closed loop pole in closed rhp if \(-\frac{1}{K} \in \mathbb{R} \)
Application to:

For $e(t) = a \sin(wt+\theta)$ view $g$ as approximated by:

Because the complex number $N(a)$ is the transfer function between the input sinusoid to $e(t)$ and the first harmonic in the output of $n$.

Also, assume $N(a)$ works for $e(t)$ of the form $ae^{zt}\sin(wt+\theta)$ [as well as the case we just did - for $e(t)$ of the form $a\sin(wt+\theta)$]

Then, growth or decay of $e(t)$ is predictable from position of $-\frac{1}{k} = -\frac{1}{N(a)}$ with respect to $g(j\omega)$-locus.

Eg. if $-\frac{1}{N(a)}$ is here, growing $e(t)$ (if $-\frac{1}{N(a)}$ is here)
Hence can predict \( e(t) \) behavior:

\[ e(0) = 0 \quad \text{and} \quad p = 0 \]

![](image)

- \( -\frac{1}{N(a)} \) locus

Start here as \( e(0) = 0 \); corresponds to \( a = 0 \)

Initially oscillation grows as \( \frac{1}{N(a)} \epsilon \ll 1 \)

\( \Rightarrow a \) increases

\( \Rightarrow -\frac{1}{N(a)} \) moves to \( b \), but stops at \( b \) as 

\( e(t) \) and hence \( a \) decreases on left of \( b \)

\( \Rightarrow \) sustained oscillation with \( a, w \) values those at \( b \).

\( \Rightarrow \) means "implies, more or less"

\[ # \quad \text{Since:} \]

\[ \frac{-1}{N(a)} \epsilon \ll 1 \Rightarrow \text{growing amplitude } a \text{ of assumed sinusoid} \]

\[ \frac{-1}{N(a)} \text{ unshaded area} \Rightarrow \text{decaying amplitude } a \text{ of assumed sinusoid } e. \]
WARNING: describing function analysis based on many approximations ⇒ predictions not necessarily correct

Can redo theory for:

\[ e(t) = \sum_{i=1}^{r} \alpha_i \sin (w_i t + \Theta_i) \]

⇒ multiple-input describing function

for which, can be shown: if a closed loop oscillation exists, it can be predicted for some finite \( r \),

and for which:

error bounds can be obtained.