Problem 1

We first determine the equilibria:

\[ 0 = \dot{x}_1 = x_2 \]
\[ 0 = \dot{x}_2 = x_1(1 - x_2^2) \]

and so the equilibria are at \((0, 0), (\pm 1, 0)\). The Jacobian of the dynamical system writes

\[ Df = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 + 2x_1x_2 & x_1^2 - \delta \end{bmatrix} \]

Evaluated at \((0, 0)\), the Jacobian becomes

\[ Df = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \]

with characteristic equation

\[ \lambda^2 + \delta\lambda - 1 = 0 \]

and it is easily shown that \((0, 0)\) is a saddle irrespective of the value of \(\delta\).

At the equilibria \((\pm 1, 0)\), the Jacobian is

\[ Df = \begin{bmatrix} 0 & 1 \\ -2 & 1 - \delta \end{bmatrix} \]

with characteristic equation

\[ \lambda^2 + (\delta - 1)\lambda + 2 = 0. \]

The eigenvalues are found to be

\[ \lambda_{1,2} = \frac{1 - \delta \pm \sqrt{(1 - \delta)^2 - 8}}{2} \]

The stability regions are

- \(\delta < 1 - 2\sqrt{2} \Rightarrow \) Unstable node
- \(\delta = 1 - 2\sqrt{2} \Rightarrow \) Unstable improper node
- \(1 - 2\sqrt{2} < \delta < 1 \Rightarrow \) Unstable focus
- \(\delta = 1 \Rightarrow \) Center in linearized system
\begin{itemize}
  \item $1 < \delta < 1 + 2\sqrt{2} \Rightarrow$ Stable focus
  \item $\delta = 1 + 2\sqrt{2} \Rightarrow$ Stable improper node
  \item $\delta > 1 + 2\sqrt{2} \Rightarrow$ Stable node
\end{itemize}

The divergence of the vector field is

$$\nabla^T f(x) = x_1^2 - \delta.$$

By Bendixson’s Theorem, we can say that no closed orbits exist in the regions

\begin{itemize}
  \item $\{(x_1, x_2) \mid x_1 < -\sqrt{\delta}\}$
  \item $\{(x_1, x_2) \mid -\sqrt{\delta} < x_1 < \sqrt{\delta}\}$
  \item $\{(x_1, x_2) \mid x_1 > \sqrt{\delta}\}$
\end{itemize}

However, there can exist across the regions.

See Figures 1 - 6 for the effect of varying $\delta$ on the phase portraits.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Problem 1, $\delta = 0.5$}
\end{figure}

**Problem 2**

The differential equation in state-space form is

$$
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
  x_2 \\
  1 - 2\sin(x_1)
\end{bmatrix}
$$

(3)
Figure 2: Problem 1, $\delta = 0.7$

Figure 3: Problem 1, $\delta = 0.9$
Figure 4: Problem 1, $\delta = 1.0$

Figure 5: Problem 1, $\delta = 1.2$
and so the equilibria are found to be at $(\pi/6 + 2k\pi, 0)$ and $(5\pi/6 + 2k\pi, 0)$, $k \in \mathbb{N}$. The Jacobian is

$$Df = \begin{bmatrix} 0 & 1 \\ -2\cos(x) & 0 \end{bmatrix} \quad (4)$$

which, evaluated at the equilibria, yield the eigenvalues $\lambda_{1,2} = \pm 3^{1/4}$ at $(5\pi/6 + 2k\pi, 0)$ (saddle) and $\lambda_{1,2} = \pm 3^{1/4}j$ at $(\pi/6 + 2k\pi, 0)$, which does not allow to make any statements about the behavior of the nonlinear system around this equilibrium.

To find a first integral, we note

$$\frac{dV}{dt} = \frac{dV}{dx_1} \frac{dx_1}{dt} + \frac{dV}{dx_2} \frac{dx_2}{dt} = 0$$

and so we seek to find $V$ such that $\frac{\partial V}{\partial x_1} = -\dot{x}_2 = 2\sin(x_1) - 1$ and $\frac{\partial V}{\partial x_2} = \dot{x}_1 = x_2$. Integrating these partial derivatives with respect to $x_1$ and $x_2$, respectively, yields

$$V(x_1, x_2) = \frac{1}{2} x_2^2 - 2\cos(x_1) - x_1 + c, \quad c \in \mathbb{R} \quad (5)$$

To analyze the behavior of the nonlinear system around $(\pi/6 + 2k\pi, 0)$, investigate $V$ evaluated at $(\pi/6 + 2k\pi + \varepsilon_1, \varepsilon_2)$ for $\varepsilon_1, \varepsilon_2 \to 0$. Set $V$ to some arbitrary constant $C$:

$$\frac{1}{2} \varepsilon_2^2 - 2\cos(\pi/6 + \varepsilon_1) - (\pi/6 + \varepsilon_1) = C$$

$$\frac{1}{2} \varepsilon_2^2 - 2(\cos(\pi/6)\cos(\varepsilon_1) - \sin(\pi/6)\sin(\varepsilon_1)) = C$$

$$\frac{1}{2} \varepsilon_2^2 - \sqrt{3}\cos(\varepsilon_1) + \sin(\varepsilon_1) - \varepsilon_1 = C$$

$$\frac{1}{2} \varepsilon_2^2 - \sqrt{3}(1 - 2\sin^2(\varepsilon_1/2)) + \sin(\varepsilon_1) - \varepsilon_1 = C$$
Approximating $\sin(x) = x$ for “small” $x$, we obtain
\[
\frac{1}{2} \varepsilon_2 - \sqrt{3} + 2\sqrt{3}\varepsilon_2^2/4 = 0
\]
\[
\varepsilon_2^2 + \sqrt{3} \varepsilon_2 = 2c.
\]
As we approach the equilibria, the level curves of $V$ are ellipses (closed orbits, centers).

See Figure 7 for a phase portrait.

![Phase Portrait](image)

Figure 7: Phase Portrait, Problem 2

**Problem 3**

The vector field in state space form is
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
x_1 - x_3^2
\end{bmatrix}
\]

The equilibria are at $(\pm 1, 0)$ with Jacobian
\[
Df =
\begin{bmatrix}
0 & 1 \\
1 - 3x_1^2 & 0
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-2 & 0
\end{bmatrix}
\]
and so the eigenvalues are $\pm \sqrt{2} i$. The Hartman-Grobman Theorem does not apply here, and so we cannot make any definitive statements on the dynamic behavior of the nonlinear system around $(\pm 1, 0)$ without further analysis. Using the same technique employed in Problem 2 to find a first integral, we find that
\[
H(x_1, x_2) = \frac{1}{4} x_1^4 - \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 = 0 + c, \quad c \in \mathbb{R}
\]
The Lie-Derivative of $\dot{H}$ along the trajectories of the system reads

$$\dot{H} = \frac{\partial H}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial H}{\partial x_2} \frac{dx_2}{dt} = \frac{\partial H}{\partial x_1} \frac{\partial H}{\partial x_2} - \frac{\partial H}{\partial x_2} \frac{\partial H}{\partial x_1} = 0.$$  \hspace{1cm} (7)

Therefore, a contour of constant $H$ corresponds to a trajectory of the phase portrait. Due to the symmetry of the vector field, we evaluate $H$ at $(1 + \varepsilon_1, \varepsilon_2)$ for $\varepsilon_1, \varepsilon_2 \rightarrow 0$, which also describes the behavior around the equilibrium $(-1, 0)$:

$$H(x_1, x_2) = \frac{1}{4}(1 + \varepsilon_1)^4 + \frac{1}{2} \varepsilon_2^2 - \frac{1}{2}(1 + \varepsilon_1)^2$$

$$= \varepsilon_1^2 + \frac{1}{2} \varepsilon_2^2 + \text{h.o.t.} = c.$$  

Hence we have ellipses (closed orbits) around the equilibria.

**Problem 4**

We first find the equilibria:

$$\dot{x}_1 = 0 \Rightarrow x_1^2 - x_1 x_2 = 0 \Rightarrow x_1(x_1 - x_2) = 0$$

$$\dot{x}_2 = 0 \Rightarrow x_1^2 - x_2 = 0$$

Thus, the equilibria are at $(0, 0)$ and $(1, 1)$.

To show that the $x_2$-axis is invariant and the slope $dx_2/dx_1$ is infinite on this line, note that $x_1 = 0$ on the $x_2$-axis. Now

$$\frac{dx_2}{dx_1} = \frac{x_1^2 - x_2}{x_1^2 - x_1 x_2}$$  \hspace{1cm} (8)

and we note that for (8) $\rightarrow \infty$ as $x_1 \rightarrow 0$ or $x_1 \rightarrow x_2$. On the $x_2$-axis, $\dot{x}_1 = x_1^2 - x_1 x_2 = 0$, and so it is invariant.

Isoclines for different values of $c$ are found by manipulating (8):

- $c = 0 \Rightarrow x_2 = x_1^2$, the standard parabola.
- $c = 0.5 \Rightarrow x_2 = \frac{x_1^2}{2 - x_1}$
- $c = 1 \Rightarrow x_2 = 0$ or $x_1 = 0$
- $c = 2 \Rightarrow x_2 = \frac{x_1^2}{2x_1 - 1}$

For a sketch of these curves, see Figure 8.

Figures 9 and 10 show the phase portrait of the system.
Figure 8: Problem 4, Isoclines for different $c$
Figure 9: Problem 4, Phase Portrait
\dot{x}_2 = -x_2 + x_1^2, \quad \text{when} \quad x_2 > x_1^2, \quad \dot{x}_2 < 0

We can use this information to determine the direction of the trajectory.