Problem 1: Magnetic Levitation System.

Figure 1 shows a magnetic suspension system, in which a ball of magnetic material of mass $m$ is suspended in mid-air by means of an electromagnet whose current $i$ is controlled by feedback from the optically measured ball position $y$. Magnetic levitation systems are used in gyroscopes, accelerometers, and fast trains.

Defining as state variables: $x_1 := y$, the vertical (downward) position of the mass, $x_2 := \dot{y}$, and $x_3 := i$, and letting the control input $u := v$, the input voltage, the state equations are given by:

\begin{align}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{g}{m} - \frac{k}{m} x_2 - \frac{L_0 a x_3^2}{2m(a + x_1)^2} \\
\dot{x}_3 &= \frac{1}{L(x_1)} \left( -Rx_3 + \frac{L_0 a x_2 x_3}{(a + x_1)^2} + u \right)
\end{align}

where $g = 9.81 \text{ m/sec}^2$ is the acceleration due to gravity, $k = 0.001 \text{ N/m/sec}$ is a viscous friction coefficient, $R = 10\Omega$ is the series resistance of the coil, $m = 0.01 \text{ kg}$, $a = 0.05 \text{ m}$, $L_0 = 0.01 \text{ H}$, and

\begin{equation}
L(x_1) = L_1 + \frac{L_0}{1 + \frac{x_1}{a}}
\end{equation}

with $L_1 = 0.02 \text{ H}$ is the inductance of the electromagnet. Assume that the full state may be measured as shown.

Suppose that it is desired to balance the ball at a desired position $y_R = 0.05 \text{ m}$.

(a) Find the steady state values $i_{ss}$ and $v_{ss}$ of $i$ and $v$ respectively, which are necessary to maintain such balance.

(b) Show that the equilibrium point obtained by taking $u = v_{ss}$ is unstable.

(c) Using feedback linearization, design a state feedback control law to stabilize the ball at desired position $y_R = 0.05$. 

(d) Discuss how the domain of attraction of the desired equilibrium using this control law would compare to that using a standard linear control law, computed from a Jacobian linearization of the maglev system.

**Problem 2: Conventional Take-Off and Landing (CTOL) Aircraft Control.** Consider the dynamics of a two dimensional aircraft in normal aerodynamic flight as shown in the Figure (2). The horizontal and vertical axes are respectively the $x, z$ axes and $\theta$ is the pitch angle made by the horizontal aircraft (body) axis with the $x$ axis. The two inputs available to the aircraft are the horizontal thrust $u_1$ (in the aircraft body frame) and the pitch moment $u_2$. To be able to write the simplest rigid body equations for the airframe we need to define the *flight path angle* $\gamma$ as $\tan^{-1}(\dot{z}/\dot{x})$ and the *angle of attack* $\alpha = \theta - \gamma$. The lift and drag forces $L, D$ are given (in the wind frame) by the following with $a_L, a_D$ being the lift and drag coefficients:

$$
L = a_L(\dot{x}^2 + \dot{z}^2)(1 + c\alpha) \\
D = a_D(\dot{x}^2 + \dot{z}^2)
$$

The equations are given by:

$$
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} = R(\theta) \begin{bmatrix}
R^T(\alpha) \\
-L
\end{bmatrix} \begin{bmatrix}
-D \\
L
\end{bmatrix} + \begin{bmatrix}
u_1 \\
u_2 - \epsilon u_2
\end{bmatrix} + \begin{bmatrix}
0 \\
-1
\end{bmatrix}
$$

(6)

and in addition we have that

$$
\dot{\theta} = u_2
$$

(7)

as a model for the generation of the pitching moment. The equations are highly stylized though accurate in form with all masses and moments of inertia, and forces due to gravity normalized away. Also in the equation (6) above the matrices $R(\alpha), R(\theta)$ are rotation matrices of the form

$$
R(\theta) = \begin{bmatrix}
cos(\theta) \\
sin(\theta)
\end{bmatrix} \begin{bmatrix}
-cos(\theta) \\
sin(\theta)
\end{bmatrix}
$$

(8)

Physically the presence of $\epsilon$ models the fact that the process of generating pitching moment results in a small downward force. Linearize the system from the inputs $u_1, u_2$ to the outputs $y_1 = x, y_2 = z$. Give the linearizing control law. Assume that the zero dynamics of the aircraft correspond to the aircraft flying at a constant altitude and constant horizontal velocity $\dot{x} = v$ with $v$ constant (trim). Derive the zero dynamics of the system. Draw (you can use Matlab) a phase portrait of these zero dynamics for some "normalized" representative values $\epsilon = .1, v = .17, a_D = 2, a_L = 30, c = 6$. Can you predict from the zero dynamics what would actually happen if you used this control law for tracking for $\epsilon$ values greater than 0? Can you suggest an alternative approximate control law (no need to derive it in functional form, just suggest how you would derive it)?

Simulate the system with control law to show that the system behaves as expected.

*Hint: In your calculations, you may wish to keep $D, L$ in symbolic form in your calculations as long as you can.*
Problem 3: MIMO Sliding Mode Control.

(a) Consider the uncertain nonlinear TITO system:
\[
\begin{align*}
\dot{x} &= f(x) + \Delta f(x) + g_1(x)u_1 + g_2(x)u_2 + \Delta g_1(x)u_1 + \Delta g_2(x)u_2 \\
y_1 &= h_1(x) \\
y_2 &= h_2(x)
\end{align*}
\]
Assume that the nominal system (meaning the system resulting from setting \(\Delta f(x), \Delta g_i(x)\) to zero) has well defined vector relative degree \((\gamma_1, \gamma_2)\). Further, assume that the uncertainties satisfy the generalized matching conditions:
\[
\begin{align*}
L_{\Delta f} L_i h_j &\equiv 0 \quad \text{for} \quad 0 \leq i \leq \gamma_j - 2 \\
L_{\Delta g_i} L_i h_j &\equiv 0 \quad \text{for} \quad 0 \leq i \leq \gamma_j - 1
\end{align*}
\]
for \(j = 1, 2\) and \(k = 1, 2\).

Construct two sliding surfaces and a sliding control law which gives robust tracking.

(b) Now, consider the following equations for a continuous stirred tank reactor involving the concentrations of three reagents \(A, B,\) and \(C\) whose fractions in the tank are given by \(x_1, x_2, x_3\). The sensed temperature is given by \(x_4\).
\[
\begin{align*}
\dot{x}_1 &= -r_1 + (c_1 - x_1)u_1 \\
\dot{x}_2 &= r_1 - r_2 + (c_2 - x_2)u_1 \\
\dot{x}_3 &= r_2 + (c_3 - x_3)u_1 \\
\dot{x}_4 &= \alpha_1 r_1 + \alpha_2 r_2 + (c_4 - x_4)u_1 + u_2
\end{align*}
\]
Here \(c_1, c_2, c_3, c_4, \alpha_1, \alpha_2\) are constants and
\[
\begin{align*}
r_1 &= e^{(\beta_1 - \beta_2)x_2} \\
r_2 &= e^{(\beta_3 - \beta_4)x_2}
\end{align*}
\]
for constants \(\beta_i\). Now suppose the outputs of this system are \(y_1 = x_1\) and \(y_2 = x_4\); and that the dynamics of the system are subject to uncertainties of the type \(\Delta f, \Delta g_1, \Delta g_2\) as in part (a) above. Derive conditions on the kinds of these uncertainties that can be overcome using MIMO sliding mode control.