Problem 1: Regular Pendulum.
Consider the pendulum equation:
\[ \ddot{\theta} = -a \sin \theta - b \dot{\theta} + cT \]  
where \( a > 0, \quad b \geq 0, \quad c > 0, \) and \( \theta \) is the angle that the rod makes with the vertical axis, \( T \) is the torque applied to the pendulum. We will assume that the torque is the control input. Suppose we would like to stabilize the pendulum at an angle \( \theta = \delta \). For the pendulum to maintain equilibrium at \( \theta = \delta \), the torque must have a steady state component \( T_{ss} \) which satisfies
\[ 0 = -a \sin \delta + cT_{ss}. \]
Choose as state variables
\[ x_1 = \theta - \delta \quad \text{and} \quad x_2 = \dot{\theta}, \]
and the control as
\[ u = T - T_{ss}. \]
Assume \( a = c = 10, \delta = \pi/4, \) and \( b = 0. \)

(a) Using Jacobian Linearization, linearize the system about the origin. Now, using linear state feedback with gains \( K = [k_1 \ k_2] \) with \( k_1 = 2.5 \) and \( k_2 = 1 \) around this linear system, show that the resulting closed loop system is locally asymptotically stable.

(b) Find a Lyapunov function for the closed loop system in (a), and use it to estimate the region of attraction.

Problem 2: Domain of Attraction II.
Consider the damped nonlinear oscillator
\[ \ddot{y} + 2\zeta \dot{y} + (1 - y)y = 0 \]  
where \( \zeta \) is a constant, with \( 0 < \zeta < 1. \)

(a) Using the state variable definition, \( x_1 = y, \ x_2 = \frac{\dot{y} + \zeta y}{\gamma}, \) where \( \gamma = \sqrt{1 - \zeta^2} \), find an estimate of the domain of attraction of the equilibrium at the origin \((x_1, x_2) = (0, 0)\), using the indirect method of Lyapunov. Where is the other equilibrium point and what is its stability type?

(b) Now, obtain an estimate of the domain of attraction of the origin, using the Lyapunov function \( V = y^2 - \frac{\zeta}{2}y^3 + y^2. \) Compare this with the domain that you computed in part (a).

Consider the following system on \( \mathbb{R}^n \):
\[ \dot{x} = f(x, t) + g(x, t) \]  
Assume that \( f(0, t) = g(0, t) \equiv 0. \) Further, assume that:

1. \( 0 \) is an exponentially stable equilibrium point of \( \dot{x} = f(x, t) \)
2. \( |g(x, t)| \leq \mu |x| \quad \forall x \in \mathbb{R}^n \)

Show that \( 0 \) is an exponentially stable equilibrium point of system (3) for \( \mu \) small enough. The moral of this exercise is that exponential stability is robust!

Problem 4: Generalized Exponential Stability Theorem.
Prove the following modification of the exponential stability theorem of the text: If there exists a function
\( v(x, t) \) and some constants \( h, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) such that for all \( x \in B_h, t \geq 0 \)

\[
\begin{align*}
\alpha_1 |x|^2 &\leq v(x, t) \leq \alpha_2 |x|^2 \\
\frac{dv(x, t)}{dt} &\leq 0 \\
\int_t^{t+\delta} \frac{dv(x, t)}{dt} dt &\leq -\alpha_3 |x(t)|^2
\end{align*}
\] (4)

Then \( x(t) \) converges exponentially to zero.

In these equation the derivative of \( v \) is understood to be along the trajectories of the differential equation

\[ \dot{x} = f(x, t) \]

This problem is very interesting in that it allows for a decrease in \( v(x, t) \) proportional to the norm squared of \( x(t) \) over a window of length \( \Delta \) rather than instantaneously.