Problem 1

Consider the actuated single and double pendulums depicted in Figures 1 (a) and (b), respectively. The equations of motion for the single pendulum are

$$\ddot{\theta} = \frac{g}{l} \sin(\theta) + \frac{1}{m} u$$  \hspace{1cm} (1)

where $\theta$ is the angle the arms make with the vertical, $g > 0$ is the gravitational constant $m > l > 0$ is the mass of the arm, $l > 0$ its length and $u \in \mathbb{R}$ the torque applied about the joint.

For the double pendulum, the equations of motion are given by

$$M(\theta_1, \theta_2) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + H(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$  \hspace{1cm} (2)

where $M$ is the mass matrix, $H$ collects the gravity and Coriolis terms, and $u_1, u_2 \in \mathbb{R}$ are the torques applied to the two joints. The masses of the two arms $m_1, m_2 > 0$, their lengths $l_1, l_2 > 0$ and the gravitational constant are contained in $M$ and $H$.  

For the problem, first put both systems into state space form. Then, construct a CLF for each system and use it to construct a control law which stabilizes the system to the origin.
Note: This problem is meant to provide some hints for how to attack Problem 3 in Homework 8. During the discussion, I’ll work through the problem for the single pendulum in detail, but only provide some hints for how you can extend the solution to the double pendulum.

CLFs and the Linear Quadratic Regulator

Consider the linear time-invariant system
\[ \dot{x} = Ax + Bu, \quad x(0) = x_0 \]  
with state \( x \in \mathbb{R}^n \), control \( u \in \mathbb{R}^m \) and \( A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m} \) where the pair \((A, B)\) is stabilizable. Recall that we say that \((A, B)\) is stabilizable if there exists a control law \( u(x) \) which makes (3) asymptotically stable.

Next, let’s consider the infinite horizon optimization problem
\[ \inf_{u(\cdot)} \int_0^\infty x^T(t)Qx(t) + u^T(t) \cdot u(t) dt \]  
where \( x(\cdot) \) solves (3), \( Q = Q^T \in \mathbb{R}^{n \times n} \) is positive definite and \( u(\cdot) \) is the control we apply to the system. The term \( x^T(t)Qx(t) \) in the integral encourages a choice of control which drives the state to the origin, while \( u^T(t) \cdot u(t) \) forces us to balance off the amount of control effort we exert to meet this objective. Note that if \( x(t) \) does not tend to 0 as \( t \to \infty \) then the cost in (4) will be infinite. This means that our choice of control \( u(\cdot) \) must stabilize the system. Since we assumed \((A, B)\) is stabilize, we know that there must exist some choice of control law which meets this objective.

It can be shown that the solution to our optimization problem is the state feedback law
\[ u(t) = -Kx(t) \]  
where
\[ K = B^TP \]  
and \( P = P^T \in \mathbb{R}^{n \times n} \) solves the Riccati equation
\[ A^TP + PA - PBB^TP = -Q. \]

We won’t take the time to prove that this is the optimal control law (see e.g. [1] for more details), but we will show how we can use a CLF based controller to recover this "optimal" trade-off between performance and control effort.

In particular, let’s consider the candidate CLF
\[ V(x) = x^TPx \]  
whose time derivative is given by
\[ \dot{v}(x) = x^TP[Ax + Bu] + [Ax + Bu]^TPx = x^T(A^TP + PA)x + 2x^TPBu \]
Letting \( \alpha_1(x) = x^T(A^TP + PA)x \) and \( \alpha_2(x) = 2x^TPB \), consider the controller given by

\[
u_{\text{opt}}(x) = \arg \min_{u \in \mathbb{R}^m} \alpha_1(x) + \alpha_2(x)u + u^T \cdot u
\]

which is slightly different than the min-norm controller from the lecture notes:

\[
u_{\text{mn}}(x) = \arg \min_{u \in \mathbb{R}^m} u^T \cdot u
\]

s.t. \( \alpha_1(x) + \alpha_2(x)u < 0 \)

While \( u_{\text{mn}} \) selects the smallest control which decreases \( V(x) \) at every point in the state-space, \( u_{\text{opt}} \) trades off how much we decrease \( V \) with how much control effort we exert. Let’s try to compute \( u_{\text{opt}} \) in closed-from. Taking the derivative of the cost with respect to \( u \) and equating it to zero we obtain:

\[
\frac{\partial}{\partial u} (\alpha_1(x) + \alpha_2(x)u + u^T \cdot u) = \alpha_2(x)^T + 2u = 0
\]

Since the above stationary conditions have a unique solution

\[
u = -\frac{1}{2} \alpha_2(x) = -B^T Px
\]

and \( \alpha_1(x) + \alpha_2(x)u + u^T \cdot u \) is a convex function of \( u \), we conclude that \( u_{\text{opt}}(x) = -B^T Px \) as desired. Finally, if we plug this control back into (9) we obtain

\[
\dot{v}(x) = x^T(A^TP + PA)x + 2x^TPBu = x^T(A^TP + PA)x - 2x^TPBB^TPx
\]

\[
= x^T(A^TP + PA - PBB^TP)x - x^TPBB^Tx \leq -x^TQx. \tag{16}
\]

Thus, we see that the closed loop system meets Lyapunov’s conditions for exponential stability. It’s pretty remarkable that we were able to recover the globally optimal solution for the optimization problem (4) using the CLF-based point-wise optimization problem in (10). While we have investigated this phenomena for linear systems, similar results can be shown to hold for optimal control problems formulated over nonlinear systems [2].

References
