Introduction to Backstepping

Let’s consider the system

\[
\begin{align*}
\dot{x} &= f(x) + g(x)z \\
\dot{z} &= u
\end{align*}
\]

where \( x \in \mathbb{R}^n, z \in \mathbb{R} \) and \( u \in \mathbb{R} \). Our goal is to drive the state \( x \) to the origin asymptotically, and we will assume that \( f(0) = 0 \). Here, we can view \( z \) as a ”virtual input” to the differential equation for \( x \). Indeed, suppose that we know a ”feedback law” \( \phi(x) \) such that

\[
\dot{x} = f(x) + g(x)\phi(x)
\]

is asymptotically stable and that we have a Lyapunov function for the reduced order system \( V(x) : \mathbb{R}^n \to \mathbb{R} \) such that

\[
\frac{d}{dx}V(x)\left[f(x) + g(x)\phi(x)\right] \leq -W(x)
\]

with \( W \) a p.d. 

Adding and subtracting \( g(x)\phi(x) \) from the right hand side of (1) we obtain

\[
\begin{align*}
\dot{x} &= [f(x) + g(x)\phi(x)] + g(x)\left[z - \phi(x)\right] \\
\dot{z} &= u
\end{align*}
\]

Thus, at least intuitively we can see that if we drive \( z(t) \to \phi(x(t)) \) then we will have \( x(t) \to 0 \). Defining the new coordinate

\[
\eta = z - \phi(x)
\]

the dynamics of the system can be represented as

\[
\begin{align*}
\dot{x} &= f(x) + g(x)\phi(x) + g(x)\eta \\
\dot{\eta} &= u - \frac{d}{dx}\phi(x)[f(x) + g(x)\phi(x) + g(x)\eta] \\
&= \alpha(x, \eta)
\end{align*}
\]
Now, if we apply the control law
\[ u = \alpha(x, \eta) + v \] (6)
the closed-loop dynamics for the system become
\[ \dot{x} = f(x) + g(x)\phi(x) + g(x)\eta \]
\[ \dot{\eta} = v \]

Next, consider the candidate CLF
\[ \hat{V}(x, \eta) = V(x) + \frac{1}{2}z^2 \] (7)
whose time derivative is
\[ \dot{\hat{V}}(x, \eta) = \frac{d}{dx}V(x)[f(x) + g(x)\phi(x)] + \frac{d}{dx}V(x)g(x)\eta + \eta v \] (8)
Thus if we set
\[ v = -\frac{d}{dx}V(x)g(x) - k\eta \] (9)
where \( k > 0 \)
\[ \dot{\hat{V}}(x, \eta) = -W(x) - k\eta^2 \] (10)
and \( \dot{\hat{V}}(x, \eta) \) will be p.d demonstrating that the system is globally asymptotically stable.

**Single Integrator Example**

Let’s consider the system
\[ \dot{x} = -x^3 + x + z \]
\[ \dot{z} = u \] (11)

By inspection, we see that if we use \( z \) to ‘cancel out’ the linear term in \( \dot{x} = -x^2 + x \) then we will have \( x \rightarrow 0 \) as desired. Concretely, if we select \( \phi(x) = -x \) and follow the above design procedure we can stabilize \( x \) to the origin. In particular note that our ”target dynamics”
\[ \dot{x} = -x^3 + x + \phi(x) = -x^3 \] (12)
are asymptotically stable. Since this system is scalar it’s straightforward to come up with a Lyapunov function (try it!).

Let’s compare this approach to feedback linearization. If we take the output for (11) to be \( y = x \) we have
\[ \dot{y} = -x^3 + x + z \]
\[ \ddot{y} = (1 - 3x^2)[x^2 - x + z] + u \]
The linearizing controller is

\[ u = -(1 - 3x^2)[x^3 - x + z] + v \] (13)

which then gives us \( \ddot{y} = v \). Note that the feedback linearizing controller destroys the "useful" nonlinear term \(-x^2\). Thus, while calculating a feedback linearizing controller for a system may be more constructive, you may end up using more energy to stabilize the system.

**Problem 1**

Consider the system

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 - x_3 \\
\dot{x}_2 &= -x_1x_3 - x_2 + u \\
\dot{x}_3 &= -x_1 + u
\end{align*}
\]

Use feedback linearization to construct a controller which makes the origin locally asymptotically stable.

**Solution to Problem 1**

Let’s try picking the output \( y = x_2 - x_3 \). In the video recording for the Discussion, I talk through my intuition for picking this output when attacking the problem. Inspecting the differential equation for the uncontrolled state \( x_1 \), we see that if we can drive \( x_2 - x_3 \rightarrow 0 \) then \( x_1 \) will also decay to zero as time goes on. We’ll then show that by driving the output (and it’s derivative) to zero we will also drive the controlled states \( x_2 \) and \( x_3 \) to zero.

Taking the derivative of the output we have

\[ \dot{y} = \dot{x}_2 - \dot{x}_3 = -x_1x_3 - x_2 + u + x_1 - u = -x_1x_3 - x_2 + x_1. \] (14)

Since in input is canceled out in the first output we take another derivative and obtain

\[ \ddot{y} = -x_1x_3 - x_1\dot{x}_3 - \dot{x}_2 + \dot{x}_1 \]

\[ = -(x_1 + x_2 - x_3)x_3 - x_1(-x_1 + u) + x_1x_3 + x_2 - u - x_1 + x_2 - x_3 \]

\[ = \alpha(x_1, x_2, x_3) - (1 + x_1)u, \]

where \( \alpha(x_1, x_2, x_3) \) is a complicated function of the state variables. Thus, we see that when \( x_1 \neq -1 \) the control law

\[ u = \frac{-1}{1 + x_1}[\alpha(x_1, x_2, x_3)v] \] (15)

gives us

\[ \ddot{y} = v \] (16)
which can be represented as the controllable linear system
\[
\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v
\]
(17)

Since our linearizing controller is undefined when \( x_1 = 1 \), from here on we will focus our attention on a small ball near the origin where the controller is well-defined.

Now, if we pick a feedback matrix \( K \in \mathbb{R}^{1 \times 2} \) such that \( A + BK \) is Hurwitz and set \( v = K[y, \dot{y}] \) then both \( y \) and \( \dot{y} \) will go to zero exponentially quickly. That is, we use apply the controller
\[
u = -\frac{1}{1 + x_1} [-\alpha(x_1, x_2, x_3) + +K[y, \dot{y}]]
\]
(18)

Next, integrating the differential equation for \( x_1 \) we have that
\[
x(t) = e^{-t}x(0) + \int_0^t e^{t-\tau}y(\tau) d\tau.
\]
(19)

Here we see that if \( y(t) \to 0 \) exponentially quickly, then \( x(t) \) will also decay to 0 at an exponential rate (try proving this to yourself formally!). Thus, by applying our feedback linearization-based controller we have \( x_1, y, \dot{y} \to 0 \) as \( t \to 0 \). Finally, we will show that this implies that \( x_2 \) and \( x_3 \) also go to 0.

Indeed, setting the output and its derivative to zero we obtain
\[
y = x_2 - x_3 = 0 \quad \dot{y} = -x_1x_3 - x_2 + x_1.
\]
(20)

Using the above equations a little algebra can be used to show that
\[
x_1 = y = \dot{y} = 0 \implies x_1 = x_2 = x_3 = 0.
\]
(21)

Thus, since our controller drives \( x_1, y, \dot{y} \to 0 \) it also drives \( x_1, x_2, x_3 \to 0 \). Thus, we see that the variables \( x_1, y, \dot{y} \) fully encode information about the variables \( x_1, x_2, x_3 \) near the origin. In particular, this suggests choosing the change of coordinates \((x_1, x_2, x_3) \to (x_1, y, \dot{y})\) for the system near the origin, to simplify the process of constructing a stabilizing controller. While the process we’ve described above might seem very ad hoc, in coming lectures we’ll see how these ideas can be generalized.