1 Problem 1

We choose the following state space representation for our dynamics:

\[
\frac{d}{dt} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -\beta \dot{\theta} - \ell \sin(\theta) \end{bmatrix}
\]  

(1)

By inspection we see that the equilibria of the system are at points of the form

\[
x^1_e = \begin{bmatrix} n2\pi \\ 0 \end{bmatrix} \quad x^2_e = \begin{bmatrix} \pi + n2\pi \\ 0 \end{bmatrix}
\]

(2)

for any integer \( n \). For simplicity we will look at the case when \( n = 0 \). In this case we linearize the dynamics about \( x^1_e \) and obtain

\[
\dot{x} = Ax
\]

(3)

where

\[
A = Df(x^1_e) = \begin{bmatrix} 0 & 1 \\ -\frac{\ell}{g} & -\beta \end{bmatrix}
\]

(4)

Analyzing the spectrum of \( A \), we see its eigenvalues are of the form

\[
\lambda = \frac{\beta \pm \sqrt{\beta^2 - 4g\ell}}{2}
\]

(5)

In the case where \( \beta = 0 \), both of the eigenvalues are imaginary and the linearization is a center. In the case where \( \beta > 0 \) both eigenvalues will have negative real part so the origin will be asymptotically stable. Depending on the relative sizes of the physical parameters \( \beta, g \) and \( \ell \) the eigenvalues may also have an imaginary part, denoting decaying oscillations.

Next, we linearize the system about \( x^2_e \) and obtain the following dynamics matrix for the linearization:

\[
A = Df(x^2_e) = \begin{bmatrix} 0 & 1 \\ \frac{\ell}{g} & -\beta \end{bmatrix}
\]

(6)
The eigenvalues for this new matrix are
\[ \lambda = \frac{\beta \pm \sqrt{\beta^2 + 4g}}{2} \] (7)

Inspecting the equation, we see that the linearization will have positive and negative eigenvalues in both the case where \( \beta > 0 \) and \( \beta = 0 \), in which case the linearization is a saddle.

2 Problem 2

We begin by taking the Taylor expansion of the vector field around the origin:
\[ f(x) \approx f(0) - \alpha x + \beta x^2 + h.o.t. \] (8)

Here we have set \( \beta = \frac{d^2}{dx^2} f(0) \) and note that \( f(0) = 0 \). In what follows we will assume we are working close enough to the origin to neglect the higher order terms. Since the eigenvalue of the linearization (simply \( -\alpha \)) is negative, the origin of the linearization is asymptotically stable. The main idea behind this problem is to show that, at least close to the origin, the linearization dominates the higher order terms of the vector field and preserves the stability properties for the full nonlinear system.

Next, let’s try to find a ball around the origin in which the linear term dominates the quadratic term. In particular, let’s try to find a \( \delta > 0 \) such that if \( x \in B(0, \delta) \) then
\[ |\alpha x| > |\beta x^2|. \] (9)

Note that if we set \( \delta < \frac{\alpha}{|\beta|} \) then the above inequality holds.

Using this argument we see that if \( x > 0 \) and \( x \in B(0, \frac{\alpha}{|\beta|}) \) we then have that
\[ f(x) < 0 \] (10)
up to second order. Similarly, we see that if \( x < 0 \) and \( x \in B(0, \frac{\alpha}{|\beta|}) \) then
\[ f(x) > 0. \] (11)

This indicates that trajectories with initial conditions in \( B(0, \frac{\alpha}{|\beta|}) \) will tend towards the origin asymptotically for the full nonlinear system.

Next, let’s consider the case of the disturbances. For notational concision, we will denote
\[ g(x) = f(x) + d(x). \] (12)

Using the approximation from above, we can approximate
\[ g(x) \approx f(0) - \alpha x + \beta x^2 + d(x) + h.o.t. \] (13)

In the first case where we have \( ||d(x)|| < c_1 ||x||^2 \) the argument is similar to the unperturbed cases above. In particular, we want the linear term \( -\alpha x \) to simultaneously dominate the quadratic terms \( \beta x^2 \) and \( c_1 ||x^2|| \). Using an argument
following the one given above, see that the linear component will dominate any quadratic terms in (13) on the set $B(0, \frac{\alpha}{|\beta| + \delta_1})$, suggesting the trajectories which start on this set will tend towards the origin asymptotically.

In the case where the disturbance can be linear in order, we may no longer be able to guarantee that trajectories that start near the origin tend towards it asymptotically. In particular, for the linearization of $f$ to dominate the disturbance and higher order terms we need to find a $\delta > 0$ so that for each $x \in B(0, \delta)$ we have

$$|\alpha x| \geq |\beta x^2| + |c_2 x|.$$  \hspace{1cm} (14)

Clearly, in the case that $c_2 > \alpha$ we cannot construct such a neighborhood. On the other hand, in the case where $c_2 < \alpha$ we can set $\delta < \frac{\alpha - c_2}{|\beta|}$ to get the desired bound, which indicates the nonlinear system is asymptotically stable on the set $B(0, \frac{\alpha - c_2}{|\beta|})$.

3 Problem 3

We didn’t get time to go over this problem in the Discussion, so we will return to it in the future.