Theorem 1. (Hartman-Grobman) Consider the planar system
\[ \dot{x} = f(x) \] (1)
where \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) and consider its linearization about an equilibrium point \( x_0 \):
\[ \delta \dot{x} = Df(x_0) \delta x. \] (2)

Then if \( Df(x_0) \) has no eigenvalues on the \( j\omega \) axis there exists a homeomorphism (a continuous map with a continuous inverse) \( h : U \to \mathbb{R}^2 \) defined on a neighborhood \( U \) of \( x_0 \) taking the trajectories of the nonlinear system to the trajectories of the linearized system. In other words, the qualitative properties of the system are preserved under linearization, at least locally.

A few notes on the theorem:
1. There exists generalizations to HG for hyperbolic equilibria in higher dimensions.
2. A related concept is that of stable and unstable manifolds, which extends the notion of eigenvectors to the nonlinear system. Try looking them up in Sastry for more details.
3. Later in the course we’ll investigate methods which use the linearization to calculate quantitative information about equilibria (e.g. regions of attraction).

Definition 1. A closed orbit of the dynamical system \( \dot{x} = f(x) \) is the trace of a nontrivial periodic trajectory, i.e., \( \gamma \) is a closed orbit if it is not an equilibrium and there exists a finite \( T > 0 \) such \( x(t + T) = x(t) \) if \( x(t) \in \Gamma \).

Definition 2. We say that a closed orbit \( \gamma \) is a limit cycle if \( \exists x_0 \notin \Gamma \) such that \( x(t) \to \gamma \) as \( t \to \infty \) or \( t \to -\infty \).

Theorem 2. (Bendixson) Suppose that \( D \) is a simply connected set in \( \mathbb{R}^2 \) such that \( \text{div}(f) \) does not vanish identically on any neighborhoods of any points in \( D \). Then \( D \) contains no closed orbits of \( \dot{x} = f(x) \).

Note: this result does not hold in higher dimensions!

1 Problem 1

1.1
Recall that the formulas relating polar coordinates to Euclidian coordinates are
\[ x = r\cos(\theta) \quad y = r\sin(\theta) \] (3)

and
\[ r = \sqrt{x^2 + y^2} \quad \theta = \arctan \left( \frac{y}{x} \right). \] (4)

Suppose we are given smooth curves \( x(t) \) and \( y(t) \). Derive a formula for \( \dot{r}(t) \) and \( \dot{\theta}(t) \) in terms of \( x(t), y(t), \dot{x}(t) \) and \( \dot{y}(t) \).

1.2
Using the formula you found above, convert the following differential equation into polar coordinates:
\[
\begin{align*}
\dot{x} &= -y + x(1 - x^2 - y^2) \\
\dot{y} &= x + y(1 - x^2 - y^2).
\end{align*}
\]
Show that the system has a limit cycle, and find what solutions tend to it as \( t \to \infty \).

1.3
Consider the system (already in polar coordinates):
\[
\begin{align*}
\dot{r} &= (1 - r^2)(2 - r^2) \\
\dot{\theta} &= -1.
\end{align*}
\]
Find the system limit cycles and sketch the phase portrait.

2 Problem 2

2.1
Linearize the following system about the origin and classify the qualitative behavior of the linearization:
\[
\begin{align*}
\dot{x}_1 &= x_2 + x_1 x_2^2 \\
\dot{x}_2 &= -x_1 + x_1^2 x_2
\end{align*}
\]
Next, show that the nonlinear system has no closed orbits.

2.2
Show that the following system has no closed orbits using Bendixon:
\[
\begin{align*}
\dot{x}_1 &= x_1 x_2^2 \\
\dot{x}_2 &= x_1
\end{align*}
\]