Problem 1

Our goal is to show that the set

\[ B = \{(x_1, x_2) : c_1 \leq x_1^2 + x_2 \leq c_2 \} \quad (1) \]

is invariant for some values of \(0 < c_1 < c_2\). Examining the vector field, we see that its only equilibrium as at the origin. Thus, if the proposed set is positively invariant it must contain a closed orbit by Poincare-Bendixon.

Let’s try calculating the time derivative of \(V(x(t))\):

\[ \dot{V}(x) = \nabla V(x) \cdot f(x) \]
\[ = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 \]
\[ = x_1(x_1 + x_2 - x_1(x_1^2 + x_2^2)) + x_2(-2x_1 + x_2 - x_2(x_1^2 + x_2^2)) \]
\[ = (x_1^2 + x_2^2) - x_2^2(x_1^2 + x_2^2) - x_1x_2 \]
\[ = (1 - (x_1^2 + x_2^2)) \cdot (x_1^2 + x_2^2) - x_1x_2 \]

Next, we recall Young’s inequality which says that \(2|x_1 + x_2| \leq x_1^2 + x_2^2\). On the level set \(V(x) = c\) (for each \(C > 0\)) we have

\[ \dot{V}(x) = (1 - c)c - x_1x_2 \]

(2)

Using Young’s inequality we have that

\[ (1 - c)c - \frac{c}{2} < \dot{V}(x) < (1 - c)c + \frac{c}{2} \]

(3)

Now, suppose we set \(c_1 = \frac{1}{4}\). Along the level set \(V(x) = \frac{1}{4}\) we have \(\dot{V}(x) > (1 - c)c - \frac{c}{2} = 0.3125\). Since the time derivative of \(V\) is positive along this (inner) surface, trajectories cannot leave \(V\) through this level-set.

Similarly, suppose we take \(c_2 = 2\). Along the level set \(V(x) = 5\) we have \(\dot{V} < (1 - c)c + \frac{c}{2} = -1\). Since the time derivative of \(V\) is negative along this (inner) surface, trajectories cannot leave \(V\) through this level-set.

Putting these facts together, we arrive at the desired result.
Problem 2