Theorem 1. (Mean Value Theorem) Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Then, for each $x, y \in \mathbb{R}^n$ we may bound
\[
\|f(x) - f(y)\| \leq \sup_{\alpha \in [0,1]} \|Df(\alpha x + (1 - \alpha)y)\|_i \cdot \|x - y\|
\] (1)

Definition 1. We say that $x_e$ is an equilibrium of $\dot{x} = f(t, x)$ on the interval $[t_0, \infty)$ if $f(t, x_e) = 0$ for each $t \in [t_0, \infty)$. We say that the equilibrium is stable in the sense of Lyapunov if $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ such that
\[
\|x_0 - x_e\| < \delta(\epsilon) \implies \|x(t, x_0, t_0) - x_e\| < \epsilon, \forall t
\] (2)
where $x(\cdot, x_0, t_0)$ is the trajectory starting at $x_0$ at $t_0$.

Definition 2. We say that $\gamma$ is a class-K function if
1. $\gamma(0) = 0$
2. $\gamma(\|x\|) > 0$ when $\|x\| > 0$
3. $\gamma$ is strictly monotonically increasing and continuous.

Problem 1
Let $f(x): \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable functions. Let $D \subset \mathbb{R}^n$ be a compact subset. Show that there exists $L > 0$ such that
\[
\|f(x) - f(y)\| < L \|x - y\| \quad \forall x, y \in D
\] (3)

Problem 2
Newton’s method is an iterative scheme for finding zeros of the equation
\[
f(x) = 0
\] (4)
where $f: \mathbb{R} \to \mathbb{R}$ is continuously differentiable. The iteration is given by
\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}
\] (5)
First, write down the iterative sequence for the function
\[ f(x) = x^2 - c. \] (6)

Next, study the convergence of the iteration scheme using the fixed-point techniques we have introduced in class. In particular, show that if we select the initial condition for the iteration in the interval \([\sqrt{c}, \infty)\) then
\[ \lim_{n \to \infty} x_n = \sqrt{c}. \] (7)

**Problem 3**

Consider the dynamics of the inverted pendulum:
\[ \ddot{\theta} = -\sin(\theta). \] (8)

Show that the origin is stable in the sense of Lyapunov.

**Problem 4**

Consider the system
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -g(x_1)
\end{align*}
\]
where \(g\) is continuously differentiable, \(g(0) = 0\) and \(g'(z) > c\) for each \(z \in \mathbb{R}\) where \(c > 0\). Next, consider the energy function
\[
v(x_1, x_2) = \frac{1}{2} x_2^2 + \int_0^{x_1} g(z) dz \] (9)

Show that
1. the energy of this system is constant over time
2. all solutions which do not start at the origin are periodic orbits.

**Problem 5**

What can you say about the existence/uniqueness of solutions for the following systems:
\[
\begin{align*}
\dot{x} &= \sqrt{x} \quad (10) \\
\dot{x} &= 1 + \frac{1 + x^3}{1 + x^4} \\
\end{align*}
\]