Problem 1

Consider the nonlinear system

\[ \dot{x} = Ax + g(t, x) \]  \hspace{1cm} (1)

where \( A \in \mathbb{R}^{n \times n} \) is Hurwitz and we have \( x \in \mathbb{R}^n \).

1. First, consider the case where \( \|g(t, x)\|_2 \leq \gamma \|x\|_2 \) for each \( t \geq 0 \) and all \( x \in \mathbb{R}^n \). Show that the origin is globally exponentially stable for \( \gamma \) small enough.

2. Next, suppose there exists \( \delta > 0 \) such that \( \|g(t, x)\| < \delta \) for each \( t \geq 0 \) and \( x \in \mathbb{R}^n \). Note that the origin may no longer be an equilibrium point for the system. Nevertheless, show that \( x(t) \) converges to a ball containing the origin as \( t \to \infty \) and produce an estimate for the radius of this ball as a function of \( \delta \).