Problem 1

(Modification of Exponential Stability Theorem) Consider the time varying system:

\[ \dot{x} = f(x, t) \]  

where \( f(x, t) \) is Lipschitz continuous in \( x \) and piecewise continuous in \( t \), and we assume that \( f(0, t) = 0 \) for each \( t \geq 0 \). Show that if there exists a function \( v(x, t) \) and constants \( \delta, \alpha_1, \alpha_2, \alpha_3 > 0 \) such that for each \( t \geq 0 \)

\[ \alpha_1 \|x\|^2 \leq v(x, t) \leq \alpha_2 \|x\|^2 \]  

\[ \dot{v}(x(t), t) \leq 0 \]  

\[ \int_{t}^{t+\delta} \dot{v}(x(\tau), \tau) d\tau \leq -\alpha_3 \|x(t)\|^2 \]  

then the origin is a globally exponentially stable equilibrium point of the nonlinear system. In other words, we now allow \( v(x, t) \) to decrease over every interval of length \( \delta > 0 \) instead of having to decrease at every instant of time.

Problem 1 Solution

First we note that the condition involving \( \alpha_3 \) can be written as

\[ v(x(t + \delta), t + \delta) \leq v(x(t), t) - \alpha_3 \|x\|^2 \]  

Now if we use the inequality involving \( \alpha_2 \) we obtain

\[ v(x(t + \delta), t + \delta) \leq v(x(t), t) - \frac{\alpha_3}{\alpha_2} v(x(t), t) \]

\[ = (1 - \frac{\alpha_3}{\alpha_2}) v(x(t), t) \]

Now, if we let

\[ \gamma = 1 - \frac{\alpha_3}{\alpha_2} \]  

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we have
\[ \nu(x(t+n\delta), t+n\delta) \leq \gamma^n \nu(x(t), t). \] (7)

Now, using the inequalities involving \( \alpha_1 \) and \( \alpha_2 \) we have
\[ \alpha_1 \|x(n\delta)\|^2 \leq \gamma^n \nu(x(0), 0) \leq \gamma^n \alpha_2 \|x(0)\|^2 \] (8)
or
\[ |x(n\delta)| \leq \left( \frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{2}} \left( \gamma^n \right)^{\frac{1}{2}} \|x(0)\| \] (9)
thus if we let \( M = \left( \frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{2}} \) and \( \lambda = (1 - \frac{\alpha_2}{\alpha_1})^{\frac{1}{2}} \) we have
\[ \|x(n\delta)\| \leq M \lambda^n \|x(0)\| \] (10)

Now, if we let \( \mu = \frac{\log(\lambda)}{\delta} < 0 \) then for each \( t = n\delta \) for some \( n \in \mathbb{N} \)
\[ \|x(t)\| \leq M e^{\mu t} \|x(0)\|. \] (11)

**Problem 2**

(Converse Exponential Stability Theorem) Show that if \( x = 0 \) is an exponentially stable equilibrium point of the nonlinear system from the previous problem then there exists \( \nu(x, t) \) and constants \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0 \) such that
\[ \alpha_1 \|x\|^2 \leq \nu(x, t) \leq \alpha_2 \|x\|^2 \] (12)
\[ \dot{\nu}(x(t), t) \leq -\alpha_3 \|x\|^2 \] (13)
\[ \frac{\partial \nu(x, t)}{\partial x} \leq \alpha_4 \|x\| \] (14)

Hint: Let \( \phi(\tau, x, t) \) denote the solution of (1) at time \( \tau \) starting from \( x \) at time \( t \) and try
\[ \nu(x, t) = \int_t^{t+T} \|\phi(\tau, x, t)\|^2 d\tau \] (15)
where \( T \) is some large constant. The proof for this and the previous problem share some common ideas, but your actual calculations will likely be a bit different.

**Problem 2 Solution**

First, let’s begin by proving proposition 5.3 from Sastry which says that the solutions to (1) satisfies
\[ e^{-Lt} \|x(0)\| \leq \|x(t)\| \leq e^{Lt} \|x(0)\| \] (16)
where $L$ is a Lipschitz constant for $f(·, t)$. We already showed the second inequality in lecture using Bellman-Gronwall. To prove the first inequality we note that

$$\frac{d}{dt}|x| \leq |Lx|. \quad (17)$$

Now, since $f(0, t) = 0$ we have that

$$\frac{d}{dt}|x| \leq ||dx|| \leq L|x|. \quad (18)$$

Multiplying by $-1$ and reversing the inequalities we then have

$$\frac{d}{dt}|x| \geq -L|x|. \quad (19)$$

This now implies that we can underbound

$$|x(t)| \geq e^{-Lt}|x(0)| \quad (20)$$

as desired.

Next, let’s return to the proposed

$$v(x, t) = \int_t^{t+T} \|\phi(\tau, x, t)\|^2 d\tau. \quad (21)$$

From the exponential stability of the system we know that there exists $\alpha > 0$ and $M > 0$ such that

$$M|x| e^{-\alpha(\tau-t)} \geq |\phi(\tau, x, t)| \geq |x| e^{-L(\tau-t)}. \quad (22)$$

where we have used the underbound on the rate of decay for the system. Now this implies that

$$v(x, t) = \int_t^{t+T} |\phi(\tau, x, t)|^2 d\tau \geq \int_t^{t+T} e^{-2L(\tau-t)}|x|^2 = \left(1 - \frac{e^{-2LT}}{2L}\right) |x|^2 \quad (23)$$

$$v(x, t) = \int_t^{t+T} |\phi(\tau, x, t)|^2 d\tau \leq \int_t^{t+T} M^2 e^{-2\alpha(\tau-t)} |x|^2 = M^2 \left(1 - \frac{e^{-2\alpha T}}{2\alpha}\right) |x|^2 \quad (24)$$

thus we set $\alpha_1 = \left(\frac{1-e^{-2LT}}{2L}\right)$ and $\alpha_2 = M^2 \left(\frac{1-e^{-2\alpha T}}{2\alpha}\right)$.

Next, if we differentiate $v(x(t), t)$ with respect to $t$ we get

$$\frac{dv(x(t), t)}{dt} = |\phi(t+T, x, t)|^2 - |\phi(t, x, t)|^2 + \int_t^{t+T} \frac{d}{dt}|\phi(\tau, x(t), t)|^2 d\tau \quad (25)$$

But for each $\Delta t$ we have $\phi(\tau, x(t+\Delta t), t + \Delta t) = \phi(\tau, x(t), t)$, thus we conclude that $\frac{d}{dt}|\phi(\tau, x(t), t)|^2 = 0$. Thus, we have

$$\frac{dv(x(t), t)}{dt} = |\phi(t+T, x, t)|^2 - |\phi(t, x, t)|^2 \leq (Me^{-\alpha T})^2 |x|^2 - |x|^2 = -(1-M^2e^{-2\alpha T})|x|^2 \quad (26)$$

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Thus, if we pick $T > \frac{1}{\alpha} \log(M)$ and set $\alpha_3 = 1 - M^2 e^{-2\alpha T} > 0$ then we have

$$\frac{dv(x(t), t)}{dt} \leq -\alpha_3 |x(t)|^2$$

as desired.