Final Review Answers

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Problem 4

First Lyapunov Function

First, let’s derive conditions on $u_1(t)$ so that $V(x, t)$ is a valid Lyapunov function. Since we need $V(\cdot, t)$ to be positive definite and decrescent for each $t$, we clearly need $u(t) > 0$ for each $t \geq 0$.

Next, let’s try to ascertain the stability of the system. The time derivative of the Lyapunov function is given by

$$
\dot{V}(x, t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t) = -x_1^2 \frac{u'_1(t)}{u_1(t)} + \left[2x_1 \quad 2x_2 \frac{1}{u_1(t)} \right] \left[-u_1(t)x_1 - u_2(t)x_2 \right]
$$

which can be simplified to

$$
\dot{V}(x, t) = -x_2^2 \left( \frac{u'_1(t)u_1(t) + u_2(t)}{u_1(t)} \right)
$$

Thus, if for each $t \geq 0$ we additionally have that $u'_1(t)u_1(t) > u_2(t)$ then $\dot{V}(x, t) \leq 0$ and we have that the system is SISL.

In the special case that $u_1(t) = \bar{u}_1$ and $u_2(t) = \bar{u}_2$ are constant, we can apply Lasalle’s Theorem to try to ascertain the stability using $V$. Assuming that $\bar{u}_1, \bar{u}_2 > 0$ (so that $\dot{V} \leq 0$), let us define

$$
\Omega_C = \mathbb{R}^2
$$

$$
S = \{x \in \mathbb{R}^2 : x_2 = 0\}
$$

$$
M = \{0\}
$$

Inspecting the dynamics, we see that $M$ is the largest invariant set contained in $S$, thus by the global version of Lasalle’s Theorem we have that the origin is globally asymptotically stable.
Second Lyapunov function

Again, for this case we see that we need \( u_1(t) > 0 \) to obtain a valid Lyapunov function for the system. In this case the time derivative is given by
\[
\dot{V}(x,t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) = x_1^2 u_1'(t) + \left[ 2x_1 u_1(t) - 2x_2 \right] \left[ -u_1(t)x_1 - u_2(t)x_2 \right]
\]
which may be reduced to
\[
x_1^2 u_1'(t) - 2u_2(t)x_2^2
\]
Thus, we see that if \( u_1'(t) < 0 \) and \( u_2(t) > 0 \) then the system will be asymptotically stable. However, the system cannot be exponentially stable, as we must have that \( u_1'(t) \to 0 \) as \( t \to \infty \) so that \( u_1(t) \) does not become negative.

Problem 6

Note that we saw this system before in the discussion. However, this time we can only choose the output as \( y = x_1, y = x_2 \) or \( y = x_3 \). Looking at the problem, we see that if we choose \( y = x_2 \) or \( y = x_3 \) then the relative degree of the output will be 1, meaning the dimension of the zero dynamics with 2. Eyeballing the zero dynamics we would obtain after applying the linearizing controller, it’s not immediately clear whether we would be able to verify that the zero dynamics are asymptotically stable. Thus, let’s try \( y = x_1 \) and start taking time derivatives:
\[
\begin{align*}
\dot{y} &= -x_1 + x_2 - x_3 \\
\ddot{y} &= -\dot{x}_1 + \dot{x}_2 - \dot{x}_3 = x_1 - x_2 + x_3 - x_1 x_3 - x_2 + u + x_1 - u = 2x_1 - 2x_2 + x_3 - x_3 x_1 \\
\end{align*}
\]
This can be rearranged into the form
\[
\ddot{y} = \alpha(x_1, x_2, x_3) + -(1 + x_1) u
\]
Thus, we see that as long as \( 1 + x_1 \) is bounded away from zero the system will be full state linearizable and the control \( u = \frac{1}{1+x_1}(-\alpha(x_1, x_2, x_3) + v) \) gives us \( \ddot{y} = v \). From the lecture notes, we know that if we set \( v = \alpha_1 \ddot{y} + \alpha_2 \dot{y} + \alpha_3 y \) where the polynomial \( s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha \) is Hurwitz then the output and its first two derivatives will decay to zero exponentially. Thus, under this control law the system is locally exp. stable since there are no zero dynamics.

Problem 7

0.1 Part a

In order to enforce the desired dissipation rate, we need to enforce
\[
x_2 = -\alpha x_1.
\]
This suggests the sliding surface
\[ s(x) = x_2 + \alpha x_1 = 0 \] (10)

**0.2 Part b**

To get a handle on this question, let’s start by simply computing the time derivative of the Lyapunov function when we apply \( u = -Ms\text{sign}(s(x)) \):

\[ \dot{V} = s(x) \cdot \dot{s}(x) = s(x) \cdot (\dot{x}_2 + \alpha \dot{x}_1) = s(x) \cdot (-M \text{sign}(s(x)) + \Delta(x,u) + \alpha x_2) \] (11)

Next, using the bound \( \|\Delta(x,u)\| \leq \gamma \) we have that
\[ \dot{V} \leq -Ms(x)\text{sign}(s(x)) + \gamma s(x) + s(x)|x_2| \] (12)

Now, if \( M \geq \gamma + |x_2| \) then we will have that \( \dot{V} \leq 0 \), as desired. However, since \( |x_2| \) will grow as we move further away from the origin, we see that there is no finite value of \( M \) which will make the origin globally asymptotically stable (compare this to the controller in the lecture notes which uses feedback linearization and a sliding term – why is the control strategy we’re using here weaker?). We can, however, choose \( M > 0 \) so that the origin is SISL. Remember that SISL only requires that trajectories which start near the origin stay near the origin. In particular, we see that if we choose \( M > \gamma \) then on some ball containing the origin we will have \( \dot{V} < 0 \), meaning the system is locally asymptotically stable, and hence SISL. In particular, you can show that on this ball the trajectories of the system will reach the sliding surface in finite time, and then converge to the origin.

**0.3 Part c**

In practice, the controller is likely to be implemented digitally, in which case the discontinuous control law cannot be updated exactly when the trajectory reaches the sliding surface. This could cause the controller to chatter (meaning that it rapidly changes between \( u = M \) and \( u = -M \)) which can damage the actuators of the system. To overcome this, one can replace the discontinuous control law with a controller which interpolates between \( -M \) and \( +M \) along a thin band containing the sliding surface.