Problem 1

See slides from midterm review.

Problem 2

Since the problem tells us to rule out closed orbits, let’s try using Bendixon’s Theorem. This is usually a good go-to if a problem tells you to rule out limit cycles, or if you have a problem about existence of limit cycles and you’re not sure where to start. Calculating the divergence of the vector field we obtain

\[
div(f) = \frac{df_1}{dx_1} + \frac{df_2}{dx_2} = -1 + 3 = 2 \quad (1)
\]

Since this quantity is constant, non-zero and does not change sign we conclude there are no closed orbits.

Problem 3

We could notice right off the bat that on the set

\[
\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2\} \quad (2)
\]

the vector field reduces to

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1
\end{align*}
\]

which are simply the dynamics of a linear oscillator. However, suppose that we didn’t notice this. How would we go about solving this problem?

Whenever you see a vector field with a term of the form \(x_1^2 + x_2^2\) you can usually get a good feel for the dynamics of the system by calculating the Lie derivative of \(V(x) = x_1^2 + x_2^2\). Let’s try doing that here:

\[
\dot{V}(x) = x_1x_2 - x_1x_2 + x_2^2(1 - x_2^2 + x_2^2) = x_2^2(1 - x_2^2 + x_2^2) \quad (3)
\]
Since $x_2^2 > 0$, we see that if $0 < V(x) < 1$ then $\dot{V}(X) > 0$ and if $V(x) > 1$ we have $\dot{V}(x) < 0$. This implies that $V(x) \rightarrow 1$ as $t \rightarrow \infty$ so long as the initial condition is not at the origin. Since there are no equilibria on the set $V(x) = 1$, we conclude that this set must contain a limit cycle.

Problem 4

The set up for this question is a canonical example of when to use a contraction map type argument. Indeed, letting $Tx = Ax + b$ we have that

$$\|Tx - Ty\| = \|Ax - Ay\| \leq \|A\|_i \|x - y\| \quad (4)$$

where $\|A\|_i$ is the induced matrix norm with respect to whatever norm we decided to use for $\mathbb{R}^n$. Now, here we see that if $\|A\|_i < 1$ then clearly $T$ is a contraction map and our sequence converges to a unique limit. However, what if its a contraction map in one norm but not another? This is very possible, as long as we can find one norm in which $T$ is contracting we can invoke the global contraction mapping results. Since this question was somewhat abstract, this would suffice for an answer. If you were given a particular matrix, you would need to find a particular norm that its contracting in.