Problem 1
\[ \dot{x} = f(x, t) + g(x, t). \]
\[ f(0, t) = g(0, t) = 0 \]

0 is an exponentially stable equilibrium point of \( \dot{x} = f(x, t) \).

Thus, there exists \( \nu(x, t) \) such that

\[ \alpha_1 \|x\|^2 \leq \nu(x, t) \leq \alpha_2 \|x\|^2 \tag{1} \]
\[ \frac{d \nu}{dt} + \frac{\partial \nu}{\partial x} f(x, t) = \dot{\nu}(x, t) \leq -\alpha_3 \|x\|^2 \tag{2} \]
\[ \left| \frac{d \nu}{dx} \right| \leq \alpha_4 \|x\| \tag{3} \]

Now apply \( \nu(x, t) \) to the perturbed system:

\[ \dot{\nu}(x, t) = \frac{d \nu}{dt} + \frac{\partial \nu}{\partial x} f(x, t) + \frac{\partial \nu}{\partial x} g(x, t) \]
\[ \leq -\alpha_3 \|x\|^2 + \alpha_4 \|x\| \mu \|x\| \]
\[ \leq - (\alpha_3 - \alpha_4 \mu) \|x\|^2 \]

\[ \therefore \text{If } \mu < \frac{\alpha_3}{\alpha_4}, \quad -\dot{\nu}(x, t) \text{ is a pdf and the perturbed system is exp. stable at } 0 \]
Problem 2

The dynamics for the system are

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = \frac{-x_2^2 \sin(x_1) + 4 \sin(x_1)}{4 + 2 \sin^2 x_1} + \frac{2 \cos(x_1)}{2 + \sin^2 x_1} u \]

and the proposed CLF is

\[ V = x^T P x \]  \hspace{1cm} (1) \]

where

\[ P = \begin{bmatrix} 1.1167 & 0.0833 \\ 0.0833 & 0.1167 \end{bmatrix} \]  \hspace{1cm} (2) \]

As noted in the lecture notes, if \( x_1 \in (-\pi/2, \pi/2) \) then we will have

\[ L_g V(x) = 0 \implies L_f V(x) = 0, \]  \hspace{1cm} (3) \]

which implies that \( V \) is a CLF on \( \{(x_1, x_2) : x_1 \in (-\pi/2, \pi/2)\} \).

For the specific control law

\[ u = \frac{7x_1 + 5x_2}{\cos x} \]  \hspace{1cm} (4) \]

there are many ways to estimate the region of attraction. The most straightforward way is to use the provided Lyapov function and find a \( C > 0 \) such that

\[ \Omega_C : = \{x : V(x) \leq C\} \subset \{x : x_1 \in (-\pi/2, \pi/2)\} \]  \hspace{1cm} (5) \]

and \( \dot{V}(x) < 0 \ \forall x \in \Omega_C \setminus \{0\} \). There are several ways to go about doing this, such as linearizing the system about the origin and bounding the higher order terms that appear as was done previously in the lecture notes and prior homework solutions (see hw 7). Another method that many students used in they answer was to use matlab and a plotting package to determine visually a value for \( C \) which meets the desired conditions. Both of these are acceptable approaches, and a large number of different answers are possible.
Problem 3

By choosing the state \( x(x_1, x_2) = (q, \dot{q}) \in \mathbb{R}^{2n} \), the state-space realization for the system is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
-M^{-1}(x_1)H(x_1, x_2) & M^{-1}(x_1)
\end{bmatrix}
\tag{6}
\]

Next, we note that the system looks a lot like the linear system

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
0 & \frac{I}{2} & 0 & \cdots & 0 \\
0 & 0 & A & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
z_2 \\
u
\end{bmatrix}
\tag{7}
\]

which can also be represented as

\[
\dot{z} = Az + Bu
\tag{8}
\]

where \( z_1, z_2 \in \mathbb{R}^n, z = (z_1^T, z_2^T)^T \), and \( I \) is the \( n \times n \)-dimensional identity matrix. Indeed, we can rewrite the dynamics for the mechanical system as

\[
\dot{x} = Ax + B[-M^{-1}(x_1)H(x_1, x_2) + M^{-1}(x_1)u].
\tag{9}
\]

The main idea for our approach will be to design a CLF for the linear system, and then show that it is also a CLF for the nonlinear system using ideas from feedback linearization. To design a CLF for the linear system, we first observe that the pair \((A, B)\) is controllable, since \( \text{rank}[B \ AB] = 2n \). Thus, there exists a matrix \( K \in \mathbb{R}^{n \times 2n} \) such that \((A + BK)\) is Hurwitz and the closed loop system \( \dot{z} \) is exponentially stable. Letting \( \hat{A} = A + BK \), from the lecture notes, we know that there is a Lyapunov function for the linear system of the form

\[
V(z) = \frac{1}{2} z^T P z
\tag{10}
\]

where \( P = P^T \) is positive definite and the solution to the Lyapunov equation

\[
A^T P + PA = -Q
\tag{11}
\]

where \( Q = Q^T \) is also positive definite and

\[
\dot{V}(z) = z^T P (A + BK) z + [(A + BK)z]^T P z = -z^T Q z \leq \lambda_{\min}(Q) \|z\|^2.
\tag{12}
\]

We see that \( V(z) \) is a CLF for the "open-loop" linear system by observing that

\[
\min_{u \in \mathbb{R}} \nabla V(z) \cdot [Az + Bu] \leq \nabla V(z) [Az + BK z] \leq -\lambda_{\min}(Q) \|z\|^2
\tag{13}
\]

In other words, we have confirmed that \( V \) is a CLF for the linear system by observing that the control \( u = Kz \) renders \( -\dot{V}(z) \) positive definite.

Next, lets confirm that \( V \) is a CLF for the nonlinear mechanical system. When evaluating the time derivative of \( V \) along the nonlinear system we obtain

\[
\dot{V}(x) = \nabla V(x) [Ax + B (-M^{-1}(x_1)H(x_1, x_2) + M^{-1}(x_1)u)]
\tag{14}
\]
Now, can we 'cancel out' the nonlinearities of the system and enforce the same closed-loop behavior we had for the linear system on the nonlinear system? In particular, let’s try the control law

\[ u = M(x_1)[M^{-1}(x)H(x_1, x_2) + Kx] = (A + BK)x, \]

(15)

When we apply this control to the system we obtain:

\[ \dot{x} = Ax + BKx, \]

(16)
as desired, which demonstrates that \( V \) is also a CLF for the nonlinear system since

\[ \dot{V}(x) = -x^TQx \]

(17)
meaning the above control law renders \(-\dot{V}\) positive definite along the trajectories of the nonlinear system. We note that the closed-loop system is exponentially stable by the Lyapunov exponential stability theorem and our above calculations.