Problem 1

The differential equation is

\[ \ddot{x} = \frac{1}{M} (-kx - F_b(\dot{x})). \]

Define \((x_1, x_2) = (x, \dot{x})\), and so \(\dot{x}_1 = x_2\). Using the stiction characteristic, we can now solve for the equilibrium points:

\[ x_2^* = 0 \]

\[ -\frac{1}{M} (kx_1^* + F_b(x_2^*)) = 0 \]

Since \(x_2^* < b\), the lower part of the stiction curve is relevant:

\[ x_1^* = -\frac{1}{k} F_b(0) = \frac{1}{k} ((c - b)^2 + d) \]

and so the unique equilibrium is at

\[ x = \frac{1}{k} ((c - b)^2 + d) \]

\[ \dot{x} = 0 \]

For the given constants, the equilibria are:

- \(b = 1, c = 2, d = 3 \Rightarrow (x, \dot{x}) = (4/3, 0)\)
- \(b = 2, c = 2, d = 3 \Rightarrow (x, \dot{x}) = (1, 0)\)
- \(b = 2.1, c = 2, d = 3 \Rightarrow (x, \dot{x}) = (301/300, 0)\)

To analyze stability, take the Jacobian:

\[ Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{1}{M} \frac{\partial F_b}{\partial x_2} \end{bmatrix} \]  

(1)

Evaluating the Jacobian (1) at the equilibrium points yields

\[ Df \bigg|_{x_1^*, x_2^*} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{2}{M}(c - b) \end{bmatrix} \]

whose eigenvalues can be found as the roots of

\[ \lambda^2 - \frac{2}{M} (c - b) \lambda + \frac{k}{M} = 0 \]

which are found to be

\[ \lambda = \frac{1}{M} \left( c - b \pm \sqrt{(b - c)^2 - kM} \right) \]

For the given constants, the eigenvalues are
• $b = 1, c = 2 \Rightarrow \lambda = 1/3 \pm 2\sqrt{2}/3j$. This is an unstable focus.
• $b = 2, c = 2 \Rightarrow \lambda = \pm j$. This is a center.
• $b = 2.1, c = 2 \Rightarrow \lambda \approx -0.0333 \pm 0.999j$. This is a stable focus.

New (Linear) Stiction Characteristic

From Figure 2 of the problem set, it follows that $\dot{x}(0) = -2$. The equilibrium point therefore is

$$(x, \dot{x}) = (2/k, 0).$$

Evaluating the Jacobian (1) at this new equilibrium point yields

$$Df = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

whose eigenvalues are computed from the characteristic equation

$$\lambda^2 - \lambda + 1 = 0.$$

It follows that $\lambda = 1/2 \pm \sqrt{3}/2$, which indicates an unstable focus.

Intuitive Explanation of Plots

The slope of the stiction curve (lower arm) is

$$\frac{dF_b}{dy} = -2((y - b) - c) = -2((\dot{x} - b) - c).$$

• For $c > b = 1$, $\frac{dF_b}{dy} \big|_{\dot{x}} < 0$. This is a negative damping force, causing an unstable equilibrium.
• For $b = c = 2$, we have $\frac{dF_b}{dy} \big|_{\dot{x}} = 0$. Stiction curve is symmetric about $\dot{x} = 0 \Rightarrow$ closed orbits.
• For $b = 2.1, c = 2$, $\frac{dF_b}{dy} \big|_{\dot{x}} > 0$. Positive damping force leading to stability.
• In the linear stiction model, $\frac{dF_b}{dy} \big|_{\dot{x}} < 0$, which is similar to the case $c > b = 1$ for the quadratic stiction case.

Problem 2

We may assume that the mass flow and plenum pressure rise are positive, i.e. $x > 0, y > 0$. Thus $\text{sign}(x) = 1$ and the throttle characteristic is $F_\alpha(x) = x^2/\alpha^2$. We now seek the equilibrium:

$$0 = \dot{x} = B(C(x^*) - y^*)$$

$$\Rightarrow y^* = C(x^*) = \frac{1}{B} \left(x^* - F^{-1}_\alpha(y^*)\right)$$
Thus, the system is in equilibrium when \( x^* = F^{-1}_\alpha(y^*) \) or \( y^* = F_\alpha(x^*) \), which can be more succinctly expressed as \( C(x^*) = F_\alpha(x^*) \). The Jacobian evaluated at equilibrium \((x^*, y^*)\) is

\[
Df = \begin{bmatrix}
B \frac{\partial C}{\partial x}|_{x^*} & -B \\
1/B & -1/B \frac{\partial F_\alpha}{\partial x}|_{x^*}
\end{bmatrix}
\]

where we used the well-known identity

\[
dF^{-1}_\alpha = \frac{1}{dF_\alpha/dx}
\]

With \( F^{-1}_\alpha(y) = \alpha \sqrt{y} \) and \( y^* = x^2/\alpha^2 \), the characteristic equation of the Jacobian (2) becomes

\[
\lambda^2 + \left( 3B(x^* - a)(x^* - b) + \frac{\alpha^2}{2Bx^*} \right) \lambda + 3B(x^* - a)(x^* - b) + \frac{\alpha^2}{2Bx^*} + 1 = 0
\]

whose eigenvalues are

\[
\lambda = \frac{1}{2} \left[ -3B(x^* - a)(x^* - b) - \frac{\alpha^2}{2Bx^*} \pm \sqrt{\left[ 3B(x^* - a)(x^* - b) - \frac{\alpha^2}{2Bx^*} \right]^2 - 4} \right]
\]

Case \( x^* \notin (a, b) \)

Either \( x^* < a \) or \( x^* > b \). Therefore

\[
3B(x^* - a)(x^* - b) > 0
\]

\[
3B(x^* - a)(x^* - b) + \frac{\alpha^2}{2Bx^*} > 0
\]

\[
3B(x^* - a)(x^* - b) + \frac{\alpha^2}{2Bx^*} > \sqrt{\left[ 3B(x^* - a)(x^* - b) - \frac{\alpha^2}{2Bx^*} \right]^2 - 4} =: \sqrt{\Delta}
\]

Thus, we obtain the following three cases:

- \( \Delta > 0 \Rightarrow \text{stable node} \).
- \( \Delta = 0 \Rightarrow \text{(improper) stable node} \).
- \( \Delta < 0 \Rightarrow \text{stable focus} \).

Case \( x^* \in (a, b) \)

In the boundary case \( x^* = a \) or \( x^* = b \), observe that \( \frac{\partial C}{\partial x}|_{x^*} \), and therefore all coefficients of (3) are positive. By the Routh-Hurwitz Stability Criterion for systems of order 2, it follows that the eigenvalues have negative real part, and hence the system is stable.

If \( x^* \in (a, b) \) and, in addition, \( \frac{\partial C}{\partial x}|_{x^*} < \frac{\partial F_\alpha}{\partial x}|_{x^*} \) as well as \( B \sqrt{\frac{\partial C}{\partial x}|_{x^*} \frac{\partial F_\alpha}{\partial x}|_{x^*}} < 1 \), the second coefficient of (3) becomes negative, causing the equilibrium to become unstable.

Lastly, if \( x^* \in (a, b) \) and, in addition, \( \frac{\partial C}{\partial x}|_{x^*} > \frac{\partial F_\alpha}{\partial x}|_{x^*} \), then the equilibrium is a saddle.
Effect of $B$

- When the compressor speed is low, i.e. $B = 0.1$, the stable equilibrium corresponds to a rotating stall.
- For $B = 0.3$, the compressor surges as there exists a stable limit cycle around the equilibrium.
- For $B = 1$, there is a stable limit cycle with discontinuous jumps between different branches.

Bonus

For large $B$,

$$\frac{\dot{x}}{\dot{y}} = B^2 \frac{C(x) - y}{x - F_x^{-1}(y)}$$

Outside the region $y = C(x)$, the discontinuity becomes almost horizontal as $\dot{x} \gg \dot{y}$. That is, $x$ jumps discontinuously to follow along the characteristic along one branch. Once on the characteristic, $\dot{x} \approx 0$ and $y$ flows slowly due to the large value of $B$. When $y$ reaches a critical value, the branch becomes unstable and $x$ jumps discontinuously to the other branch.

Figures for Problems 1 and 2

![Figure 1: Problem 1, Quadratic Stiction, $b = 1$](image-url)
Figure 2: Problem 1, Quadratic Stiction, \( b = 2 \)

Figure 3: Problem 1, Quadratic Stiction, \( b = 2.1 \)
Figure 4: Problem 1, Linear Stiction, $b = 1$

Figure 5: Problem 1, Linear Stiction, $b = 2$
Figure 6: Problem 1, Linear Stiction, $b = 2.1$

Figure 7: Problem 2, $B = 0.1$
Figure 8: Problem 2, $B = 0.3$

Figure 9: Problem 2, $B = 1.0$
Figure 10: Problem 2, $B = 4$