EE222 - Problem Set 2 Solutions
February 27, 2020

Problem 1
(i) $\lambda_1 = 0, \lambda_2 < 0$

The system, after a similarity transformation, can be but into the Jordan form

$$\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} \tag{26}$$

The corresponding equations of motion are

$$\dot{z}_1 = 0$$
$$\dot{z}_2 = \lambda_2 z_2$$

So, the trajectories are

$$z_1(t) = z_1(0)$$
$$z_2(t) = z_2(0)e^{\lambda_2 t}$$

Thus, the line defined by $z_2 = 0$ is a stable equilibrium that is approached as $t \to \infty$. The phase portrait is plotted in Figure 8. Note that the similarity transform could map such a system onto any orientation or skewing of coordinates in state space.

![Figure 8: Phase portrait for $\lambda_1 = 0$ and $\lambda_2 < 0$.](image)

(ii) $\lambda_1 = 0, \lambda_2 = 0$

There are two subcases for this case. First, it is possible that $A$ is the zero matrix. In this case, all $x \in \mathbb{R}^2$ are equilibrium points.
For the second case, we have the Jordan form

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\] (27)

Where \(a \neq 0\). The corresponding equations of motion are

\[
\dot{z}_1 = az_2 \\
\dot{z}_2 = 0
\]

So, the trajectories are

\[
z_1(t) = z_1(0) + az_2 t \\
z_2(t) = z_2(0)
\]

Again, the line defined by \(z_2 = 0\) is an equilibrium, although it is not asymptotically approached in time. The phase portrait is plotted in Figure 9. Again, note that the similarity transform could map such a system onto any orientation or skewing of coordinates in state space.

Figure 9: Phase portrait for \(\lambda_1 = 0\) and \(\lambda_2 = 0\).
Problem 2

We first determine the equilibria:

\[ 0 = \dot{x}_1 = x_2 \]
\[ 0 = \dot{x}_2 = x_1(1 - x_1^2) \]

and so the equilibria are at \((0, 0), (\pm 1, 0)\). The Jacobian of the dynamical system writes

\[ Df = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 + 2x_1x_2 & x_1^2 - \delta \end{bmatrix} \]  

(1)

Evaluated at \((0, 0)\), the Jacobian becomes

\[ Df = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \]

with characteristic equation

\[ \lambda^2 + \delta \lambda - 1 = 0 \]

and it is easily shown that \((0, 0)\) is a saddle irrespective of the value of \(\delta\).

At the equilibria \((\pm 1, 0)\), the Jacobian is

\[ Df = \begin{bmatrix} 0 & 1 \\ -2 & 1 - \delta \end{bmatrix} \]

with characteristic equation

\[ \lambda^2 + (\delta - 1)\lambda + 2 = 0 \]

The eigenvalues are found to be

\[ \lambda_{1,2} = \frac{1 - \delta \pm \sqrt{(1 - \delta)^2 - 8}}{2} \]  

(2)

The stability regions are

- \( \delta < 1 - 2\sqrt{2} \Rightarrow \) Unstable node
- \( \delta = 1 - 2\sqrt{2} \Rightarrow \) Unstable improper node
- \( 1 - 2\sqrt{2} < \delta < 1 \Rightarrow \) Unstable focus
- \( \delta = 1 \Rightarrow \) Center in linearized system
- \( 1 < \delta < 1 + 2\sqrt{2} \Rightarrow \) Stable focus
- \( \delta = 1 + 2\sqrt{2} \Rightarrow \) Stable improper node
- \( \delta > 1 + 2\sqrt{2} \Rightarrow \) Stable node
The divergence of the vector field is

\[ \nabla^T f(x) = x_1^2 - \delta. \]

By Bendixson’s Theorem, we can say that no closed orbits exist in the regions

- \( \{(x_1, x_2) \mid x_1 < -\sqrt{\delta}\} \)
- \( \{(x_1, x_2) \mid -\sqrt{\delta} < x_1 < \sqrt{\delta}\} \)
- \( \{(x_1, x_2) \mid x_1 > \sqrt{\delta}\} \)

However, there can exist across the regions.

See Figures 1 - 6 for the effect of varying \( \delta \) on the phase portraits.

![Phase Portrait for Duffing Equation, \( \delta = 0.5 \)](image)

Figure 1: Problem 2, \( \delta = 0.5 \)
Figure 2: Problem 2, $\delta = 0.7$

Figure 3: Problem 2, $\delta = 0.9$
Figure 4: Problem 2, $\delta = 1.0$

Figure 5: Problem 2, $\delta = 1.2$
Problem 3

The differential equation in state-space form is

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
1 - 2\sin(x_1)
\end{bmatrix}
$$

and so the equilibria are found to be at \((\pi/6 + 2k\pi, 0)\) and \((5\pi/6 + 2k\pi, 0)\), \(k \in \mathbb{N}\). The Jacobian is

$$
Df =
\begin{bmatrix}
0 & 1 \\
-2\cos(x) & 0
\end{bmatrix}
$$

which, evaluated at the equilibria, yield the eigenvalues \(\lambda_{1,2} = \pm 3^{1/4}\) at \((5\pi/6 + 2k\pi, 0)\) (saddle) and \(\lambda_{1,2} = \pm 3^{1/4}i\) at \((\pi/6 + 2k\pi, 0)\), which does not allow to make any statements about the behavior of the nonlinear system around this equilibrium.

To find a first integral, we note

$$
\frac{dV}{dt} = \frac{dV}{dx_1} \frac{dx_1}{dt} + \frac{dV}{dx_2} \frac{dx_2}{dt} = 0
$$

and so we seek to find \(V\) such that \(\frac{\partial V}{\partial x_1} = -\dot{x}_2 = 2\sin(x_1) - 1\) and \(\frac{\partial V}{\partial x_2} = \dot{x}_1 = x_2\). Integrating these partial derivatives with respect to \(x_1\) and \(x_2\), respectively, yields

$$
V(x_1, x_2) = \frac{1}{2} x_2^2 - 2\cos(x_1) - x_1 + c, \quad c \in \mathbb{R}
$$
To analyze the behavior of the nonlinear system around \((\pi/6 + 2k\pi, 0)\), investigate \(V\) evaluated at \((\pi/6 + 2k\pi + \varepsilon_1, \varepsilon_2)\) for \(\varepsilon_1, \varepsilon_2 \rightarrow 0\). Set \(V\) to some arbitrary constant \(C\):

\[
\begin{align*}
\frac{1}{2}\varepsilon_2^2 - 2\cos(\pi/6 + \varepsilon_1) - (\pi/6 + \varepsilon_1) &= C \\
\frac{1}{2}\varepsilon_2^2 - 2(\cos(\pi/6)\cos(\varepsilon_1) - \sin(\pi/6)\sin(\varepsilon_1)) &= C \\
\frac{1}{2}\varepsilon_2^2 - \sqrt{3}\cos(\varepsilon_1) + \sin(\varepsilon_1) - \varepsilon_1 &= C \\
\frac{1}{2}\varepsilon_2^2 - \sqrt{3}(1 - 2\sin^2(\varepsilon_1/2)) + \sin(\varepsilon_1) - \varepsilon_1 &= C
\end{align*}
\]

Approximating \(\sin(x) = x\) for “small” \(x\), we obtain

\[
\begin{align*}
\frac{1}{2}\varepsilon_2^2 - \sqrt{3} + 2\sqrt{3}\varepsilon_1^2/4 &= 0 \\
\varepsilon_2^2 + \sqrt{3}\varepsilon_1^2 &= 2c.
\end{align*}
\]

As we approach the equilibria, the level curves of \(V\) are ellipses (closed orbits, centers). The important property to note for this problem (and also the following problem) is that since the level curves are \textit{closed}, the orbits they contain must also be closed. Since our above analysis has shown that there are no equilibria on these level sets (if we stay sufficiently close to the origin), we know that they must be closed orbits.

See Figure 7 for a phase portrait.

![Figure 7: Phase Portrait, Problem 3](image)
Problem 4

The vector field in state space form is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
x_2 \\
x_1 - x_1^3
\end{bmatrix}
\]

The equilibria are at \((\pm 1, 0)\) with Jacobian

\[
Df = \begin{bmatrix}
0 & 1 \\
1 - 3x_1^2 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-2 & 0
\end{bmatrix}
\]

and so the eigenvalues are \(\pm \sqrt{2}j\). The Hartman-Grobman Theorem does not apply here, and so we cannot make any definitive statements on the dynamic behavior of the nonlinear system around \((\pm 1, 0)\) without further analysis. Using the same technique employed in Problem 2 to find a first integral, we find that

\[
H(x_1, x_2) = \frac{1}{4}x_1^4 - \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 = 0 + c, \quad c \in \mathbb{R}
\]

(6)

The Lie-Derivative of \(\dot{H}\) along the trajectories of the system reads

\[
\dot{H} = \frac{\partial H}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial H}{\partial x_2} \frac{dx_2}{dt} = \frac{\partial H}{\partial x_1} \frac{\partial H}{\partial x_2} - \frac{\partial H}{\partial x_2} \frac{\partial H}{\partial x_1} = 0.
\]

(7)

Therefore, a contour of constant \(H\) corresponds to a trajectory of the phase portrait. Due to the symmetry of the vector field, we evaluate \(H\) at \((1 + \varepsilon_1, \varepsilon_2)\) for \(\varepsilon_1, \varepsilon_2 \to 0\), which also describes the behavior around the equilibrium \((-1, 0)\):

\[
H(x_1, x_2) = \frac{1}{4}(1 + \varepsilon_1)^4 + \frac{1}{2}\varepsilon_2^2 - \frac{1}{2}(1 + \varepsilon_1)^2
\]

\[
= \varepsilon_1^2 + \frac{1}{2}\varepsilon_2^2 + \text{h.o.t.} = c.
\]

Hence we have ellipses (closed orbits) around the equilibria.

Problem 5

We first find the equilibria:

\[
\dot{x}_1 = 0 \Rightarrow x_1^2 - x_1x_2 = 0 \Rightarrow x_1(x_1 - x_2) = 0
\]

\[
\dot{x}_2 = 0 \Rightarrow x_1^2 - x_2 = 0
\]

Thus, the equilibria are at \((0, 0)\) and \((1, 1)\).

To show that the \(x_2\)-axis is invariant and the slope \(dx_2/dx_1\) is infinite on this line, note that \(x_1 = 0\) on the \(x_2\)-axis. Now

\[
\frac{dx_2}{dx_1} = \frac{x_1^2 - x_2}{x_1^2 - x_1x_2}
\]

(8)

and we note that for (8) \(\to \infty\) as \(x_1 \to 0\) or \(x_1 \to x_2\). On the \(x_2\)-axis, \(\dot{x}_1 = x_1^2 - x_1x_2 = 0\), and so it is invariant.

Isoclines for different values of \(c\) are found by manipulating (8):
\begin{itemize}
  \item $c = 0 \Rightarrow x_2 = x_1^2$, the standard parabola.
  \item $c = 0.5 \Rightarrow x_2 = \frac{x_1^2}{2-x_1}$
  \item $c = 1 \Rightarrow x_2 = 0$ or $x_1 = 0$
  \item $c = 2 \Rightarrow x_2 = \frac{x_1^2}{2x_1-1}$
\end{itemize}

For a sketch of these curves, see Figure 8.

Figures 9 and 10 show the phase portrait of the system.
Figure 8: Problem 5, Isoclines for different $c$
Figure 9: Problem 5, Phase Portrait
Figure 10: Problem 5, Phase Portrait

\[ \dot{x}_2 = -x_2 + x_1^2, \quad \text{when} \quad x_2 > x_1^2. \quad \dot{x}_2 < 0 \]

We can use this information to determine the direction of the trajectory.