Problem 1. Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + x_2(1 - x_1^2 - 2x_2^2)
\end{align*}
\]

Prove that the annulus

\[
\frac{1}{2} \leq x_1^2 + x_2^2 \leq 1
\]

contains a closed orbit.

Problem 2: Lotka-Volterra Predator-Prey Equations [1]. Count Vito Volterra was an Italian mathematician (1860-1940), who developed a mathematical model to explain the results of a statistical study of fish populations in the Adriatic Sea. In particular, his model explains the increase in predator fish (and corresponding decrease in prey fish) which he observed during the World War I period. Volterra produced a series of models for the interaction of two or more species. Alfred J. Lotka was an American biologist and actuary who independently produced many of the same models. One of the simplest of their models takes the form

\[
\begin{align*}
\dot{x} &= ax - bxy \\
\dot{y} &= -dy + cxy
\end{align*}
\]

where \( x > 0 \) denotes the sardine (prey) population and \( y > 0 \) denotes the shark (predator) population. \( a, b, c, \) and \( d \) are all positive constants. Note that the equations model the facts that: sardines multiply faster as they increase in number; the number of sardines decreases as both the sardine and shark population increases; sharks increase in number at a rate proportional to the number of shark-sardine encounters. Study the equilibria of this system and show by simulation that for different values of \( a, b, c, \) and \( d, \) the model predicts both cyclic variations in population as well as convergence to steady state values.

Now repeat this analysis for a more complicated model, in which the sardine population saturates in the absence of sharks, and vice versa.

\[
\begin{align*}
\dot{x} &= (a - by - \lambda x)x \\
\dot{y} &= (-d + cx - \mu y)y
\end{align*}
\]

where \( \lambda, \mu > 0. \)

Problem 3: Van der Pol oscillator. The Van der Pol oscillator is one of the best known models in nonlinear systems theory, originally developed to describe the operation of an electronic valve oscillator, which depends on the existence of a region with effective negative resistance. This model is also used to describe the behavior of the pumping heart. The equation is that of a simple harmonic oscillator with nonlinear damping, written as:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \epsilon(x_2 - x_2^3)
\end{align*}
\]

(a) Determine the stability type of the origin \([x_1, x_2]^T = [0, 0]^T\) for \( \epsilon > 0. \)
(b) Fix $\epsilon = 1$. We can conjecture the existence of a limit cycle for the Van der Pol oscillator by simulating the system in Matlab. Now, I would like you to prove that a limit cycle exists. You can do this by using the following steps.

Consider the region shown in Figure 1. The labelled points $A, \ldots, J$ are defined as follows, for parameters $a, b \in \mathbb{R}$, where $b > 0, a > b$:

\[
\begin{align*}
A &= (-a, 0) \\
B &= (-a, 1) \\
C &= (-\sqrt{a^2 - b^2}, \sqrt{1 + b^2}) \\
D &= (0, \sqrt{1 + b^2}) \\
E &= (b, 1) \\
F &= (a, 0)
\end{align*}
\]

The lower half of the region is symmetric through the origin. The segments $AB$, $CD$ and $EF$ are straight line segments, $BC$ and $DE$ are circular arcs. The isocline $x_1 = x_2 - x_3^2$ is also shown, as are the two lines $x_2 = 1$ and $x_2 = -1$.

Show that the region enclosed by this boundary is positively invariant for suitable values of $a$ and $b$ (find these values). Then use a modified version of this region to show that a limit cycle exists for the Van der Pol oscillator with $\epsilon = 1$.

**Problem 4: Bifurcations in one-dimensional systems.** Sketch phase portraits of the following one-dimensional systems as $\mu$ changes.

1. $\dot{x} = \mu^2 x - x^3$
2. $\dot{x} = \mu^2 \alpha x + 2\mu x^3 - x^5$ for $\alpha = 0, 1, -0.5$.

**References**