Problem 1: Regular Pendulum.

Consider the pendulum equation:
\[ \ddot{\theta} = -a \sin \theta - b \dot{\theta} + cT \]  
(1)
where \( a > 0, b \geq 0, c > 0 \), and \( \theta \) is the angle that the rod makes with the vertical axis, \( T \) is the torque applied to the pendulum. We will assume that the torque is the control input. Suppose we would like to stabilize the pendulum at an angle \( \theta = \delta \). For the pendulum to maintain equilibrium at \( \theta = \delta \), the torque must have a steady state component \( T_{ss} \) which satisfies \( 0 = -a \sin \delta + cT_{ss} \). Choose as state variables \( x_1 = \theta - \delta \) and \( x_2 = \dot{\theta} \), and the control as \( u = T - T_{ss} \). Assume \( a = c = 10, \delta = \pi/4 \), and \( b = 0 \).

(a) Using Jacobian Linearization, linearize the system about the origin. Now, using linear state feedback with gains \( K = [k_1 k_2] \) with \( k_1 = 2.5 \) and \( k_2 = 1 \) around this linear system, show that the resulting closed loop system is locally asymptotically stable.

(b) Find a Lyapunov function for the closed loop system in (a), and use it to estimate the region of attraction.

Problem 2: Domain of Attraction II.

Consider the damped nonlinear oscillator
\[ \ddot{y} + 2\zeta \dot{y} + (1 - y)y = 0 \]  
(2)
where \( \zeta \) is a constant, with \( 0 < \zeta < 1 \).

(a) Using the state variable definition, \( x_1 = y, x_2 = \frac{\dot{y} + \zeta y}{\gamma} \), where \( \gamma = \sqrt{1 - \zeta^2} \), find an estimate of the domain of attraction of the equilibrium at the origin \( (x_1, x_2) = (0, 0) \), using the indirect method of Lyapunov. Where is the other equilibrium point and what is its stability type?

(b) Now, obtain an estimate of the domain of attraction of the origin, using the Lyapunov function \( V = y^2 - \frac{2}{3}y^3 + \dot{y}^2 \). Compare this with the domain that you computed in part (a).