Fractal Compression

- Founders: Manderb~~l~~roth and Barnsley.

- Basic idea: fixed point transformation

- X_0 is fixed point of function f if

$$f(X_0) = X_0$$

- Example: Transformation ax + b
  has a fixed point X_0 given by:

$$X_0 = aX_0 + b$$

- To transmit X_0, send a , b

  Then iterate:

$$X_0^{(n+1)} = aX_0^{(n)} + b$$

will converge regardless of initial guess.

Image Compression

- Think of Image I as array of numbers
- Find a function f such that

  $f(I) = I$

- If # of bits to transmit f is smaller than I, achieve Compression
- In practice, hard to come up with one transformation f, for the whole image.
- Divide up the image into domain and domain and Range blocks

Image Compression

- Main idea:

  - Divide up image into M x M "Range" blocks

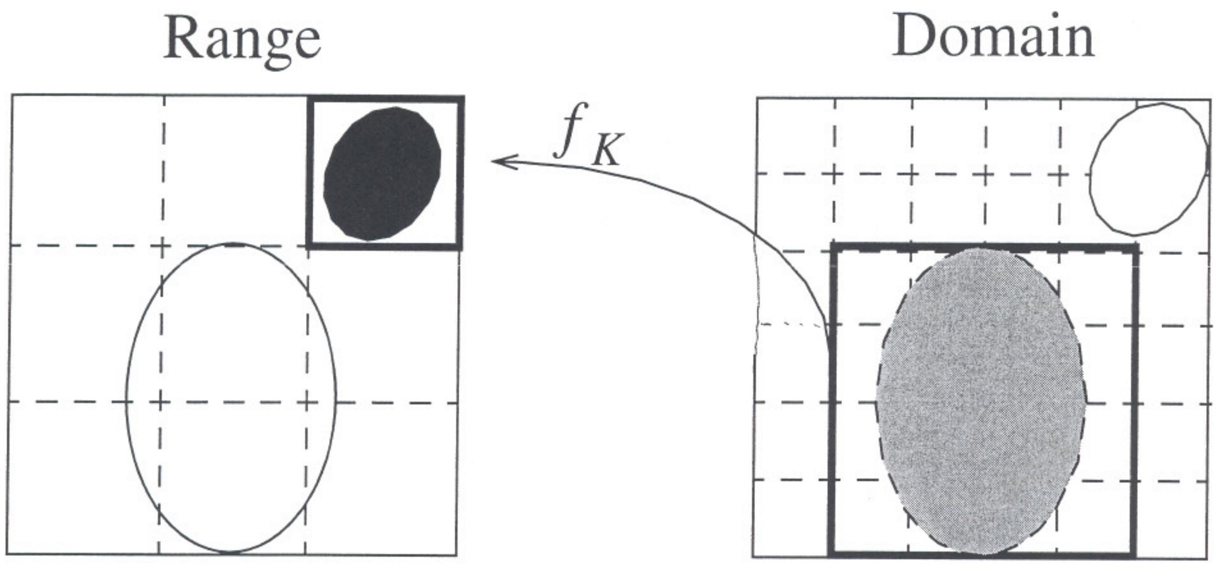
  - For each Range block find another 2 M x 2 M "Domain block" from the same image such that for some transformation f_K we get $f_K(D_K) = R_K$

    D_K = Domain block k

    R_K = Range block k

- First publicly discussed by Jacquin in 1989 thesis + 1992 paper

- Works well if there is self similarity in image.



- What should f_K do?

  - change <u>size</u> of domain block
  - change <u>orientation</u> of domain block
  - change <u>intensity</u> of pixels

- f_K consists of

  - geometric transformation : g_K
  - massic transformation : m_K

- g_K : displacement + size + intensity

- m_K : orientation + ~~intensity~~

Transformations:

- g_K : displaces + adjusts intensity

  $\longrightarrow$ easy

- m_K : $m_K(t_{ij}) = i(\alpha_K t_{ij} + \Delta_K)$

  i can be

  * Rotation by 90, 180, -90
  * Reflection about
     horizontal, vertical, diagonal
  * identity map

- Finding transformations is compute intensive

- Search through all domain blocks + all transformations to find "BEST" one

- Encoding more time than decoding

- If Image is divided *into*

  N Range blocks $\longrightarrow$ N

  transformations f_K $\quad k = 1, \ldots N$

  are its representation.

$$f = \bigcup_k f_k$$

$$\hat{I} = f(\hat{I})$$

  $\hat{I}$ is approximation to I.

- Collage theorem guarantees convergence

  to $\hat{I}$ using <u>any</u> arbitrary initial

  guess for image.

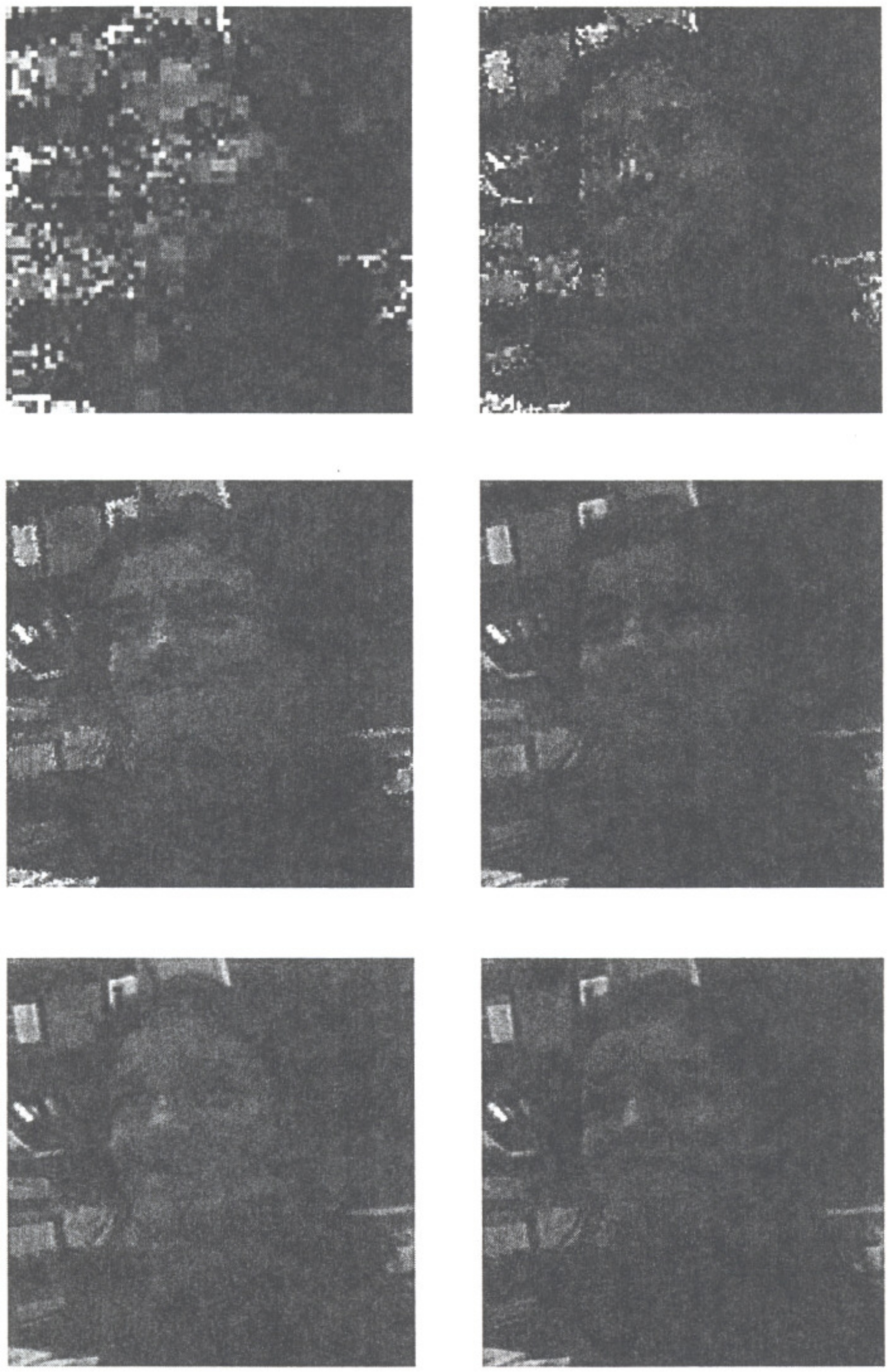


FIGURE 13.11 The first six iterations of the fractal decoding process.

VECTOR QUANTIZATION

- Let $\vec{f}$ denote N dimensional vectors constisitng of N real valued, continuous amplitude scalars.
- **Basic Idea:** Map $\vec{f}$ into L possible N dimensional reconstruction vectors $\vec{r}_i$ for $1 \leq i \leq L$.

$$\hat{\vec{f}} = VQ(\hat{f}) = \vec{r}_i \qquad \vec{f} \in C_i$$

- Define a distortion measure:

$$\begin{aligned} D &= E[(\hat{\vec{f}} - \vec{f})^T(\hat{\vec{f}} - \vec{f})] \\ &= \sum_{i=1}^{L} \int_{\vec{f}_0 \in C_i} (\vec{r}_i - \hat{\vec{f}}_0) d\vec{f}_0 \end{aligned}$$

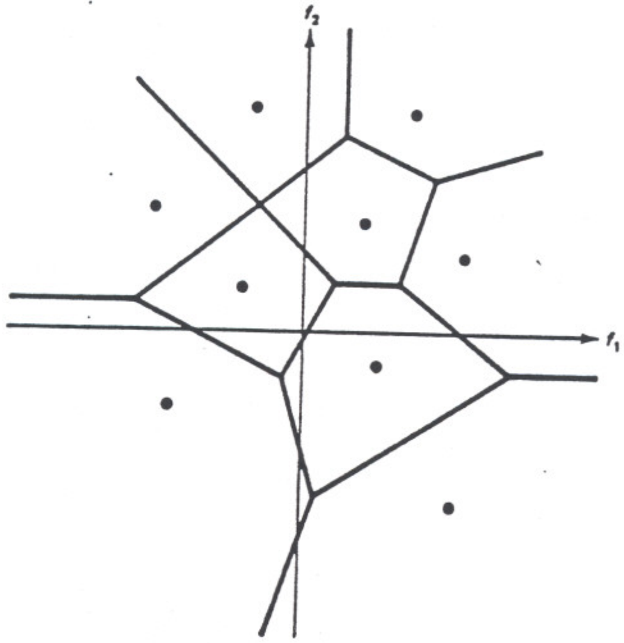


Figure 10.8 Example of vector quantization. The number of scalars in the vector is 2, and the number of reconstruction levels is 9.

Properties of Vector Quantization

- Removes linear dependency between random variables.
- Removes nonlinear dependency between random variables.
- Explits increase in dimensionality.
- Allows us to code a scalar with less than one bit.
- Computational and storage requirements are far greater than scalar quantization.

VQ Removes Linear Dependency

- Linear transformation can decorrelate linearly dependent (correlated) random variables.

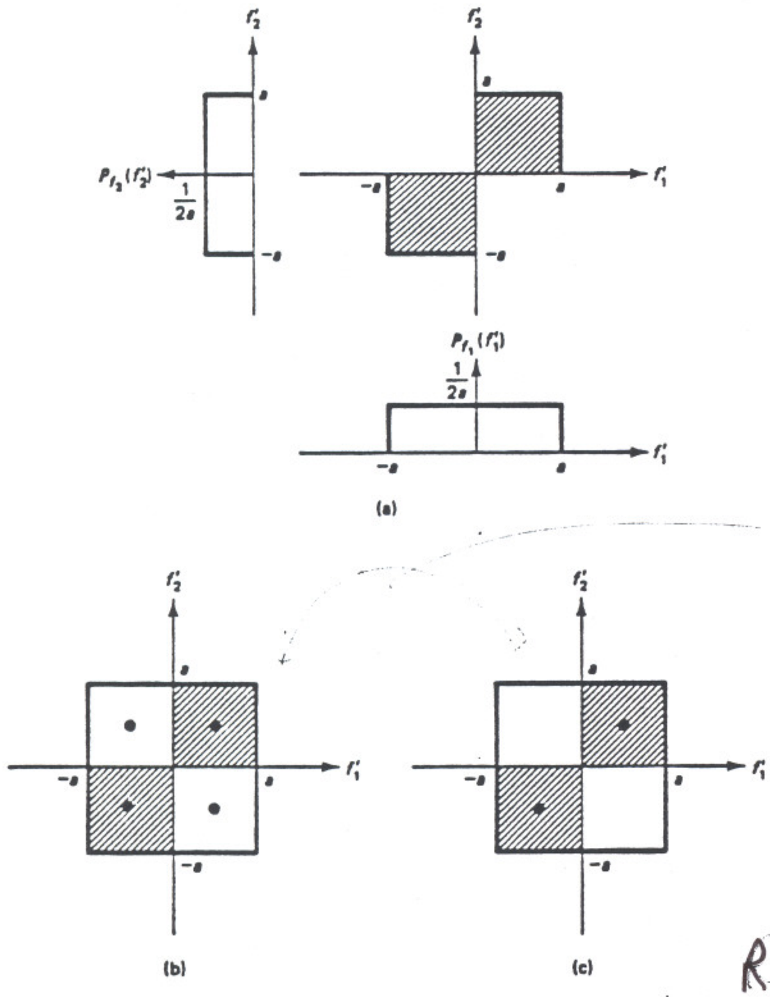


Figure 10.9 Illustration that vector quantization can exploit linear dependence of scalars in the vector. (a) Probability density function $p_{f_1,f_2}(f_1', f_2')$; (b) reconstruction levels (filled-in dots) in scalar quantization; (c) reconstruction levels (filled-in dots) in vector quantization.

Some distorTion
But reduce # of
reconstruction level

Rotate (linear Transformation) f_1 f_2 to get g_1 g_2 uncorrelated.

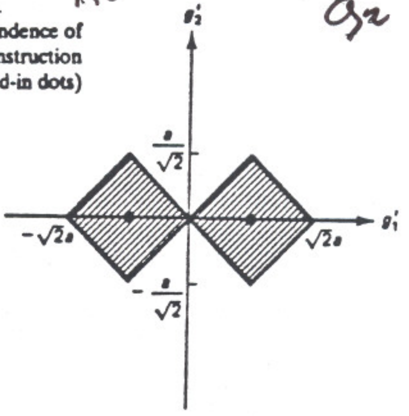
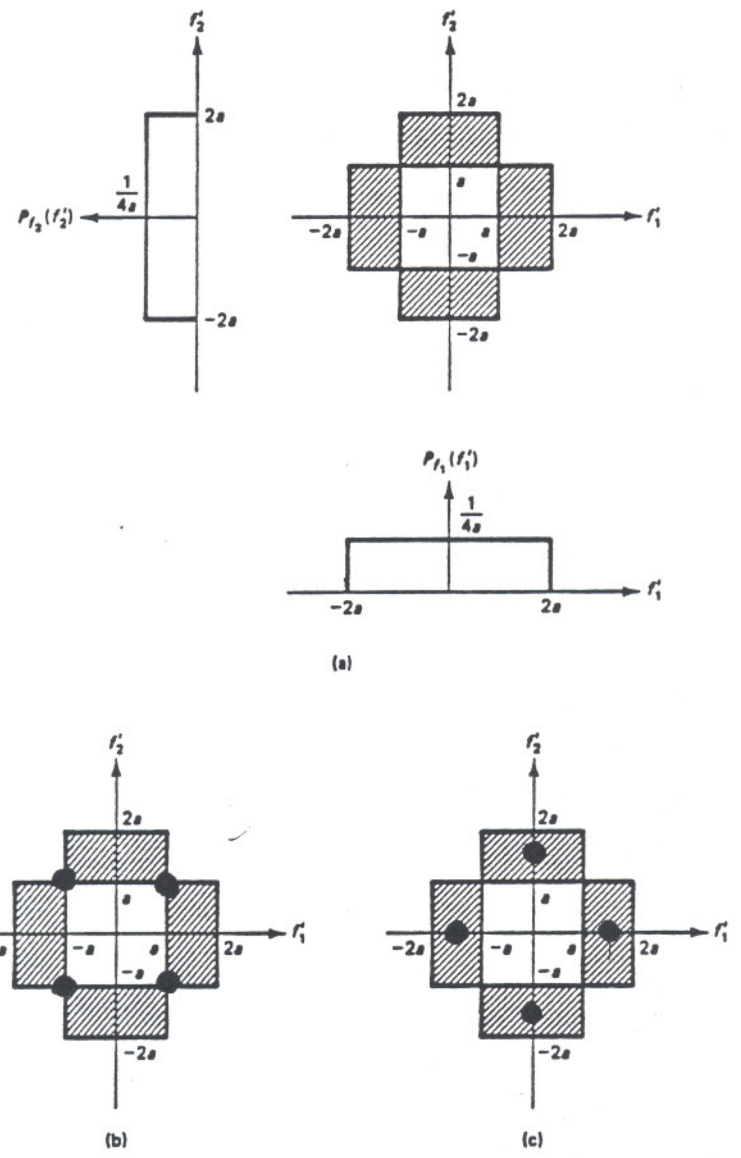


Figure 10.10 Result of eliminating linear dependence of the two scalars f_1 and f_2 in Figure 10.9 by linear transformation of f_1 and f_2.

VQ Removes Nonlinear Dependency

- Nonlinear dependence cannot be eliminated by a linear operator.



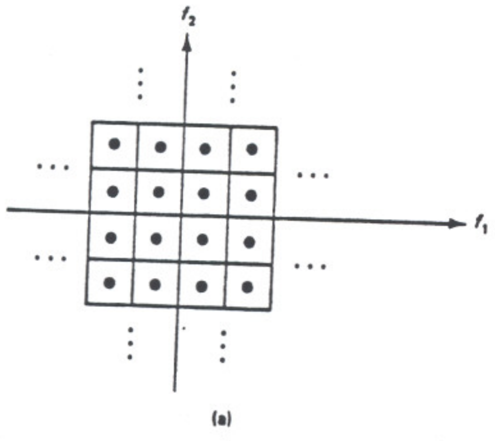
distortion
$a^2/2$

distortion
$\frac{5a^2}{12}$

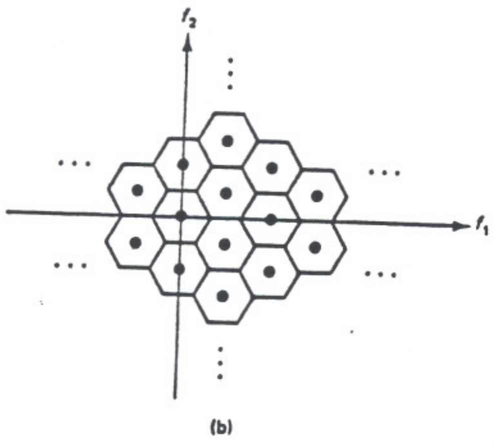
Figure 10.11 Illustration that vector quantization can exploit nonlinear dependence of scalars in the vector. (a) Probability density function $p_{f_1,f_2}(f_1', f_2')$; (b) reconstruction levels (solid dots) in scalar quantization; (c) reconstruction levels (solid dots) in vector quantization.

VQ Exploits the Increase in Dimensionality

- The mean square error due to VQ is approximately less than 4 percent than scalar quantization. with same # of reconstruction levels.



# of Recon levels is 2% lower than Scalar Quantization with same MSE.



$\Rightarrow$ # of bits per scalar with VQ < 1

look at 10.9

Figure 10.13 Illustration that vector quantization can exploit the dimensionality increase. In this case, the mean square error due to vector quantization is approximately 4% less than that due to scalar quantization. (a) Scalar quantization of f_1 and f_2; (b) vector quantization of f_1 and f_2.

scalar: 1 bit per scalar

VQ: $\frac{1}{2}$ bit per scalar

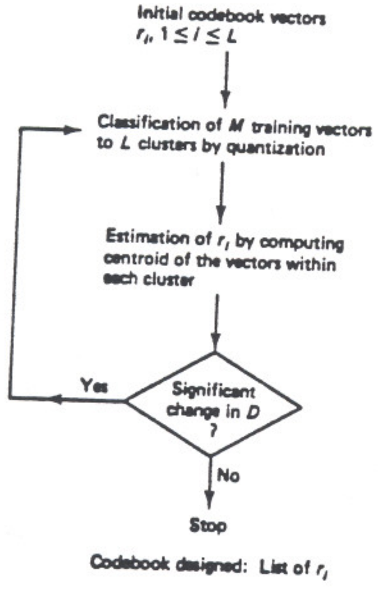
Codebook Design Algorithms

- K-means algorithm.
- Tree codebooks and binary search.
- Nearest neighbor.

Codebook Design via K-means

- Exploit the following two necessary conditions for the optimal solution:
  - For a vector $\vec{f}$ to be quantized to one of the reconstruction levels, the optimal quantizer must choose the reconstruction level $\vec{r}_i$ $1 \le j \le L$ which has the samllest distortion between $\vec{f}$ and $\vec{r}_i$. $Q(f) = r_i$ iff $d(f, r_i) < d(f, r_j)$ $j \neq i$
  - Each reconstruction level $\vec{r}_i$ must minimize the average distortion D in C_i.

    $Minimize \quad E[d(\vec{f}, \vec{r}_i) \mid \vec{f} \in C_i] \quad w.r.t. \quad \vec{r}_i$
- Find $\vec{r}_i$ and C_i iteratively → Problem: local versus global minimum → initial guess important.



Initial codebook vectors r_i, $1 \le i \le L$
Classification of M training vectors to L clusters by quantization
Estimation of r_i by computing centroid of the vectors within each cluster
Significant change in D ?
Yes
No
Stop
Codebook designed: List of r_i

Figure 10.14 Codebook design by the K-means algorithm for vector quantization.

Complexity of K-means

- M training vectors, L codewords, N dimensional, R bits per scalar.
- Complexity of Codebook design:
  - ML evaluation of distortion measure for each iteration.

$L = 2^{NR} = 2^{B}$

  - $MLN = NM2^{NR}$ additions and mults per iteration.
  - Example: N= 10, R=2 , M = 10L results in 100 trillion operation per iteration.
  - Storage: MN for training vectors, LN for reconstruction levels $\rightarrow \quad (M + 2^{NR})N$.
- Complexity of operation at the ~~receiver~~ Transmitter.
  - Storage of reconstruction levels: $N2^{NR}$ $= NL$. If $N = 10$ and $R = 2$, storage is 10 million.
  - Number of artithemetic operations $N2^{NR}$. If $N = 10$ and $R = 2$, 10 million operations per look up.

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Tree Codebook and Binary Search

- Full search is responsible for exponential growth of the number of operations at the ~~receiver~~ Txmttr. $\rightarrow$ Tree codebook.
- Let L be a power of 2.
- Basic operation of tree codebook design:
  - Use K-means to divide the N dimensional space of $\vec{f}$ into two regions.
  - Divide each of the two regions into two more regions using the K-means algorithm.
  - Repeat step 2 until there are L reconstruction levels.

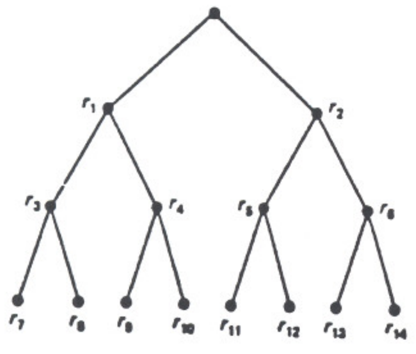

Figure 10.15 Example of a tree codebook.

M = # of Traning vectors
L = # of codewords
N = Dimension

Complexity of Tree Codebook

distortion measure evaluated twice

- Design complexity: # of stages
  - Number of arithmetic operations per iteration is $2NM\log_2 L$. For $N = 10$ and $R = 2$, the reduction factor compared to the full search is 26,000.
  - Storage: approximately the same as full search algorithm. (storing training data dom vater)
- Operation complexity at ~~receiver~~: Transmitter
  - Number of arithmetic operations Is $2N^2R$. $= 2N\log_2 L$ For $N = 10$ and $R = 2$, the reduction factor compared to the full search is 26,000.
  - Storage: The codebook must store all the intermediate reconstruction levels as well as the final reconstruction levels. $\rightarrow$ Twice as much storage needed as full search.
- Distortion of full search is slightly smaller than that of tree search.

Nearest Neighbor Design Algorithm

- Initially proposed by Equitz.
- computational complexity grows linearly with the training set.
- Find the 2 vectors closest to each other, merge them into another vector equal to their mean, repeat this process until the number of vectors is L.
- Main efficiency is achieved by partitioning the training data into a K-D tree $\rightarrow$ multiple merges at each iteration.

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Variations of VQ

- Multistage VQ reduces storage and search time.
  1. First stage a low rate VQ.
  2. Generate error by subtracting the codeword from the original.
  3. Code the error by a different VQ.
  4. Repeat steps 2 and 3.

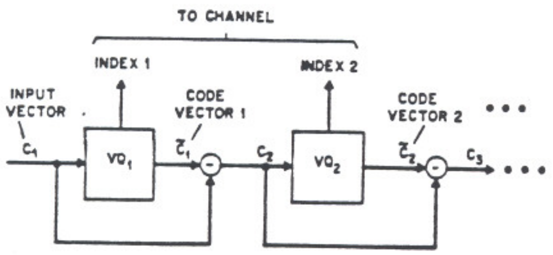

Fig. 5.5.2 Multistage Vector Quantization. At each state an error vector is computed, which is then used as the input to the next stage of VQ. The decoder merely computes a summation of the code vectors corresponding to the received indices.

- Parameter extraction techniques:
  - mean and variance of each input vector are computed and sent separately.
  - mean and variance might be coded with DPCM.

Variations of VQ (cont'd)

- Block classification:
  - Divide the blocks into several classes according to spatial activity.
  - Design a codebook for each class.
  - Overhead on transmitting the codebook is large.
- Combine prediction techniques with VQ:
  - Coded quantity is the prediction error rather than intensity values.
- VQ of color images exploits the correlation between color components.
- Typical rates: .1 to .5 bits per pixel for 4×4 pixels as vectors.

local measure, such as local image contrast. If the local measure used can be obtained from previously coded pixel intensities, then it does not have to be transmitted. If the local measure used is obtained directly from $f(n_1, n_2)$, it must be transmitted, since the receiver does not have access to the original image. Adaptive coding certainly adds complexity to an image coder, but can often significantly improve its performance.

PCM systems do not exploit the statistical dependence of neighborhood pixel intensities. One way to exploit the statistical dependence is to use methods such as DM, DPCM, and two-channel coders, where the difference between $f(n_1, n_2)$ and a prediction of $f(n_1, n_2)$ is coded. An alternate way is to use vector quantization. As we discussed in Section 10.2, vector quantization can exploit the statistical dependence of the parameters coded. Vector quantization has been considered in coding the waveform $f(n_1, n_2)$. The blocks used consist of neighborhood pixel intensities of a small size, typically 2×2, 3×3, and 4×4. Vector quantization has been primarily applied in low bit rate (below 1 bit/pixel) applications, since the computational and storage costs increase rapidly with the block size and the bit rate. Intelligible images can be reconstructed with some sacrifice in quality at bit rates below 1 bit/pixel, which is not possible with DM, DPCM, or two-channel coders with scalar quantization and uniform-length codeword assignment. For waveform coders, vector quantization is an effective way to code an image at a bit rate lower than 1 bit/pixel. Figure 10.40 illustrates the performance of vector quantization when applied to coding the waveform $f(n_1, n_2)$. Figure 10.40(a) shows an original image of 512×512 pixels. Figure 10.40(b) shows the

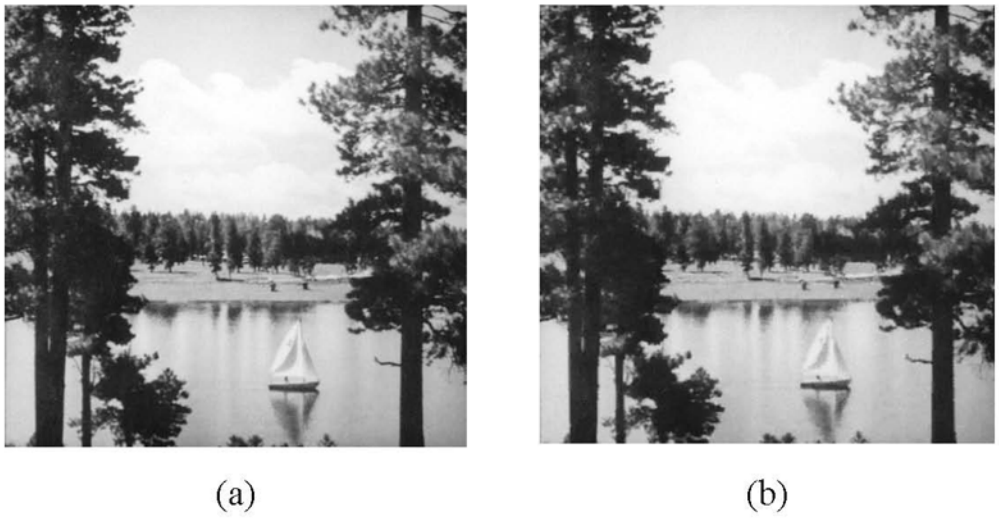

Figure 10.40 Example of an image coded by vector quantization. Courtesy of William Equitz. (a) Original image of 512 x 512 pixels; (b) coded image by vector quantization at 1/2 bit/pixel. The block size used is 4 x 4 pixels and the codebook is designed by using a variation of the K-means algorithm. NMSE = 2.7%, SNR = 15.7 dB.