Systems

- system Transform input into output

$$T[\, x(n_1, n_2) \,] = y(n_1, n_2)$$

input

output

- <u>Linear</u>: $T\{ a\, x_1(n_1, n_2) + b\, x_2(n_1, n_2) \}$

$$= a\, T\{ x_1(n_1, n_2) \} + b\, T\{ x_2(n_1, n_2) \}$$

- <u>Shift Invariance</u>:

If $T\{ x(n_1, n_2) \} = y(n_1, n_2)$

then $T[\, x(n_1 - k_1, n_2 - k_2) \,] = y(n_1 - k_1, n_2 - k_2)$

2D - LSI system.

LSI $\longrightarrow$ Impulse response.

$h(n_1, n_2)$

- enables to entirely characterize an LSI system.
- tells you output given input.

$$Y(n_1, n_2) = X \ast h$$

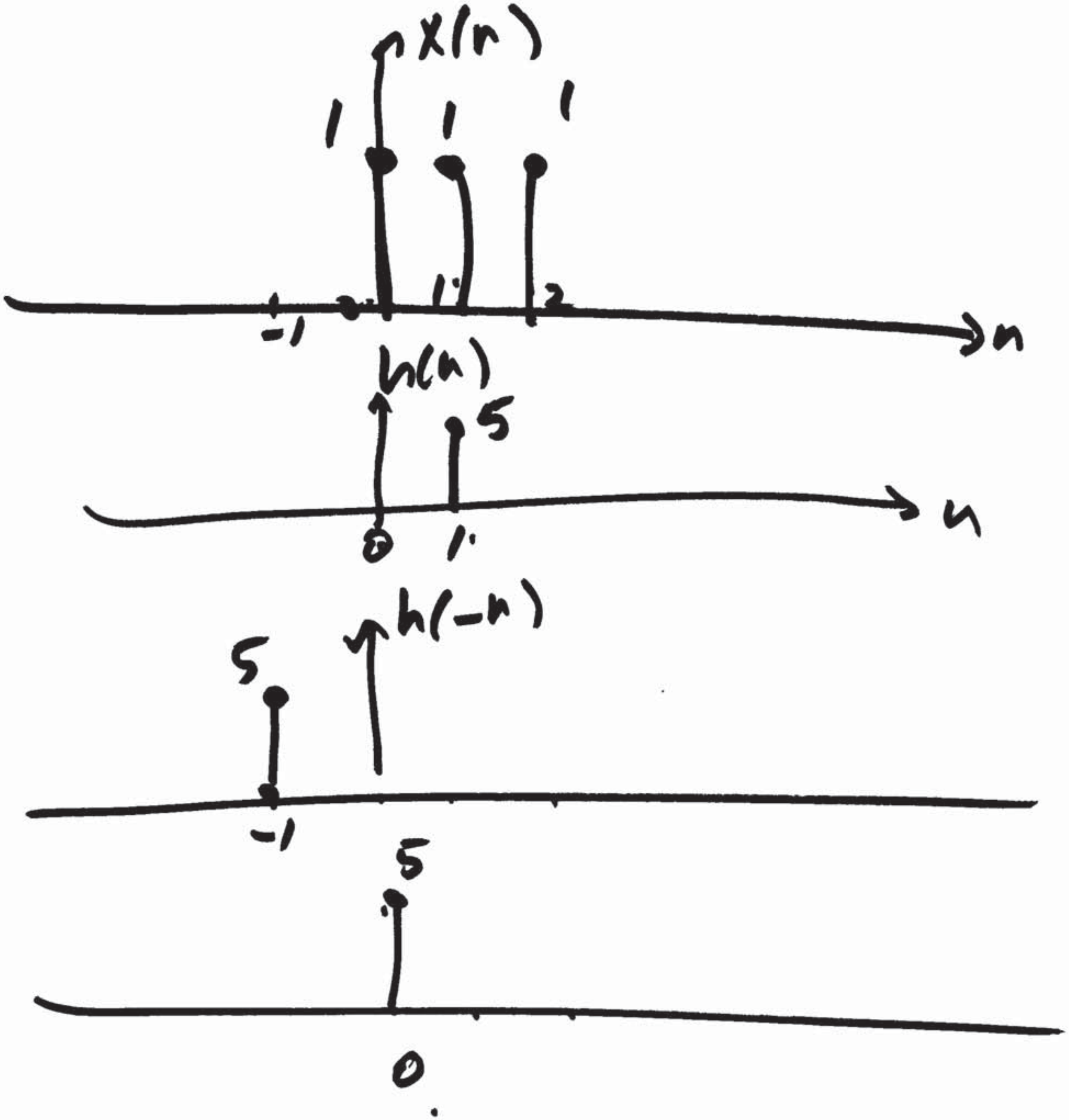
Convolution

$$y(n_1, n_2) = \sum_{k_1} \sum_{k_2} x(k_1, k_2)\, h(n_1 - k_1, n_2 - k_2)$$

2D

$$Y(n) = \sum_{k} x(k)\, h(n-k)$$

1D



$y(0) = 0$

$Y(1) = 5$

Separable LSI system

$$h(n_1,n_2) = h_1(n_1)\, h_2(n_2)$$

$x(n_1,n_2)$ $N \times N$ $\qquad x \rightarrow \boxed{h} \rightarrow y.$

h : $M \times M.$

general:

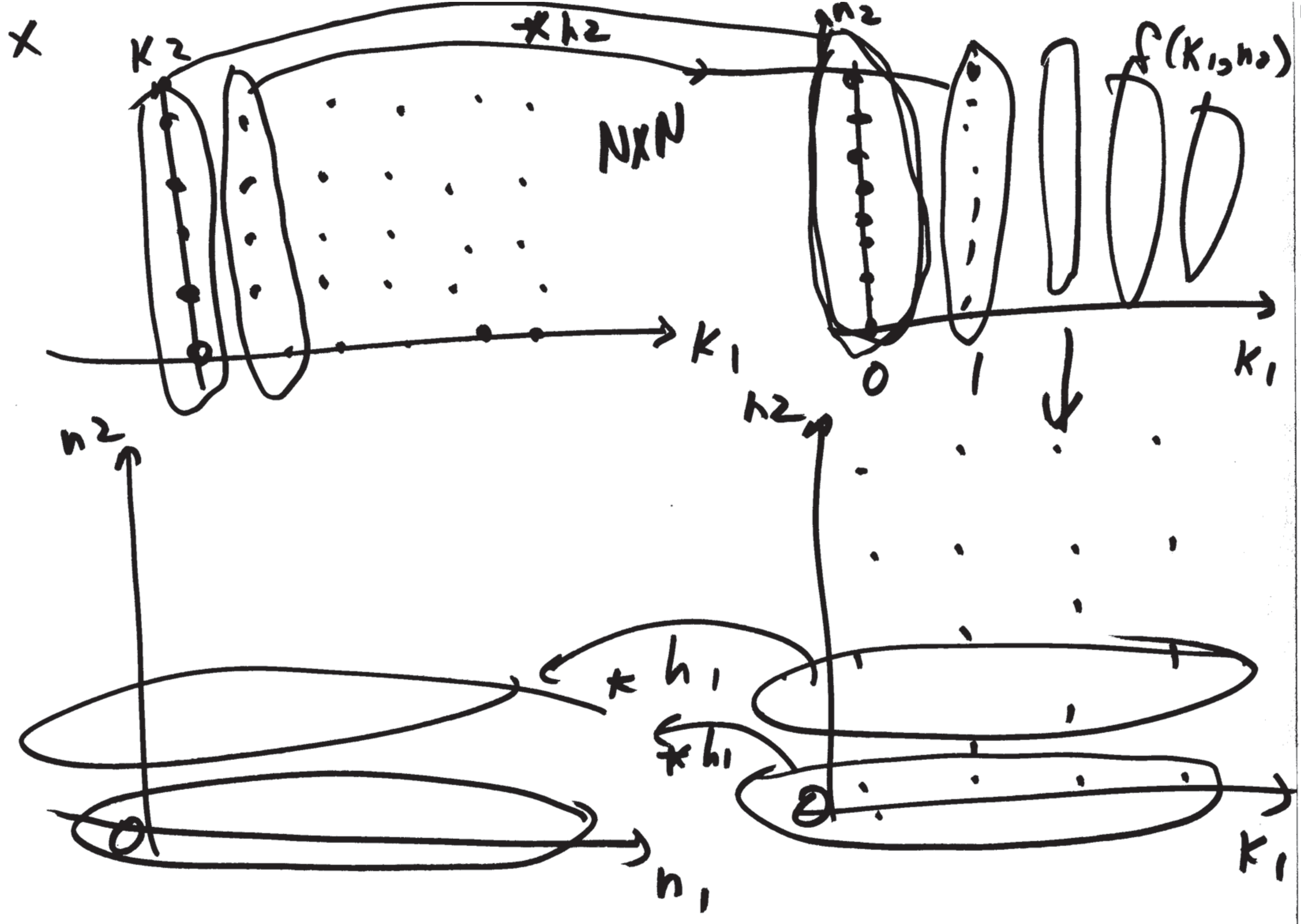
$$y(n_1,n_2) = \sum_{k_1}\sum_{k_2} x(k_1,k_2)\, h(n_1-k_1, n_2-k_2)$$

sep. sys.

$$= \sum_{k_1}\sum_{k_2} x(k_1,k_2)\, h_1(n_1-k_1)\, h_2(n_2-k_2)$$

$$= \sum_{k_1} h_1(n_1-k_1) \underbrace{\sum_{k_2} x(k_1,k_2)\, h_2(n_2-k_2)}_{f(k_1,n_2) =}$$

$x(k_1) * h_2$



X
k2
*h2
n2
NXN
f(k1,n2)
k1
0
1
k1
n2
n2
*h1
*h1
n1
k1

Summary

1. Series of 1D convolution to get $f(k_1, n_2)$:

N columns $\rightarrow$ N convolution
NM oper / convolut

$N^2 M$ op to build f

(2) Series of 1-D conv to get $g(n_1, n_2)$

N rows $\rightarrow$ N convolution
NM op / conval

$N^2 M$

Total $2 N^2 M$.

LSI sytem

* impulse response. $\rightarrow h(n_1, n_2)$
* stability of LSI system.
* BIBO stability $\rightarrow$

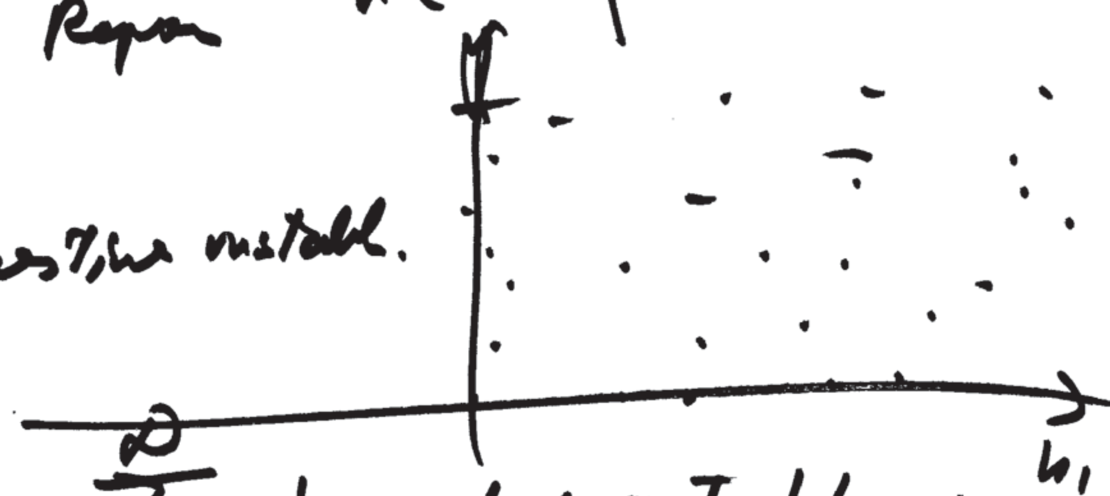
  bounded input, bounded output.
* Can show, necess + suff. cond. $h(n_1, n_2)$ to be BIBO stable.

$$\sum_{n_1 = -\infty}^{+\infty} \sum_{n_2} |h(n_1, n_2)| < \infty$$

Finite impulse response
(FIR).

$\Longrightarrow$ Always stable.

Infinite impulse Respon
(IIR)
some stable, sometime unstable.



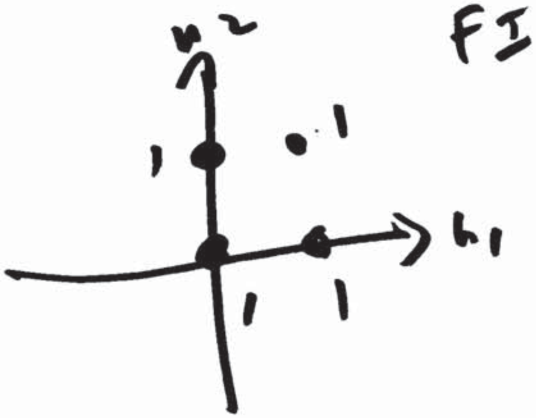
$\sum_{n=1}^{\infty} \frac{1}{n}$ blows up $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does not blow up.

$$h(n_1, n_2) \xrightarrow{Z.T.} \sum_{n_1} \sum_{n_2} h(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2}$$

FIR

$$= H(z_1, z_2) = \text{2D polynomial in } z_1 \,\&\, z_2$$

FIR



$$H(z_1, z_2) = 1 + z_1^{-1} + z_2^{-1} + z_1^{-1} z_2^{-1}$$

2D polynomial in z_1^{-1} and in z_2^{-1}

2.D.F.T.

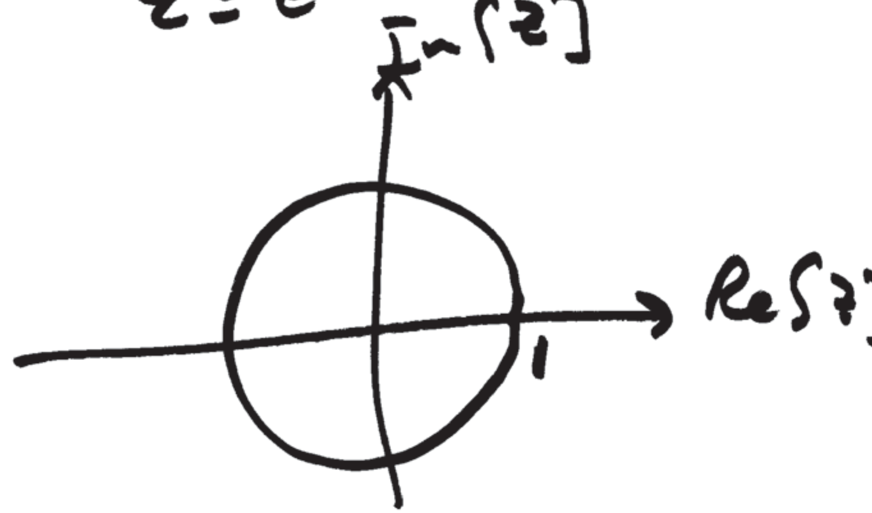
$$H(\omega_1, \omega_2) = \sum_{n_1} \sum_{n_2} h(n_1, n_2)\, e^{-j\omega_1 n_1}\, e^{-j\omega_2 n_2}$$

$$e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix} LSI \\ h(n) \end{matrix}} \longrightarrow \underbrace{H(\omega_0)}_{\text{not a fn of } n.} e^{j\omega_0 n}$$

$$H(\omega_0) = \Big[H(\omega)\Big]_{\omega = \omega_0} = \left[\sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n}\right]_{\omega = \omega_0}$$

$$\left.\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} \\ H(z) &= \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n} \end{aligned}\right\} \Rightarrow$$

$$H(\omega) = \left[H(z) \right]_{z=e^{j\omega}}$$



$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \longrightarrow h(n) = \ldots$$

ROC: outside circle

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

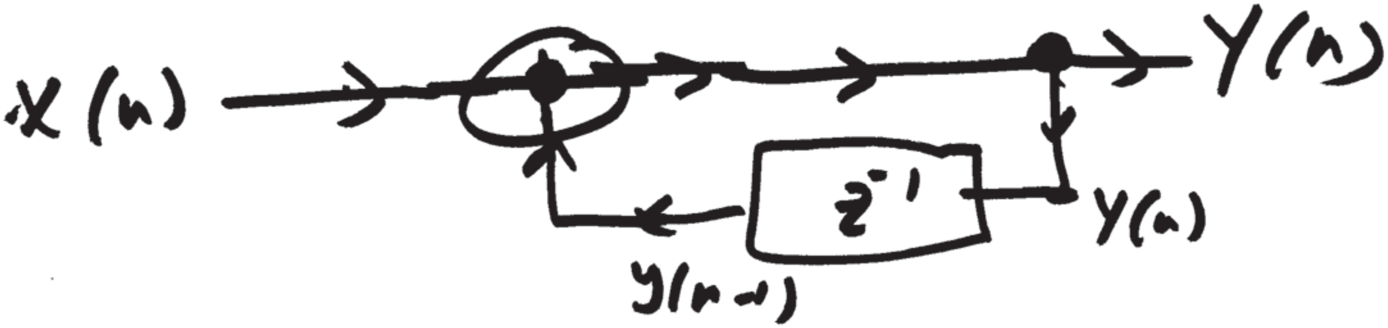
$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

BIBO

$$Y(z)\left[1-\frac{1}{2}z^{-1}\right] = X(z).$$

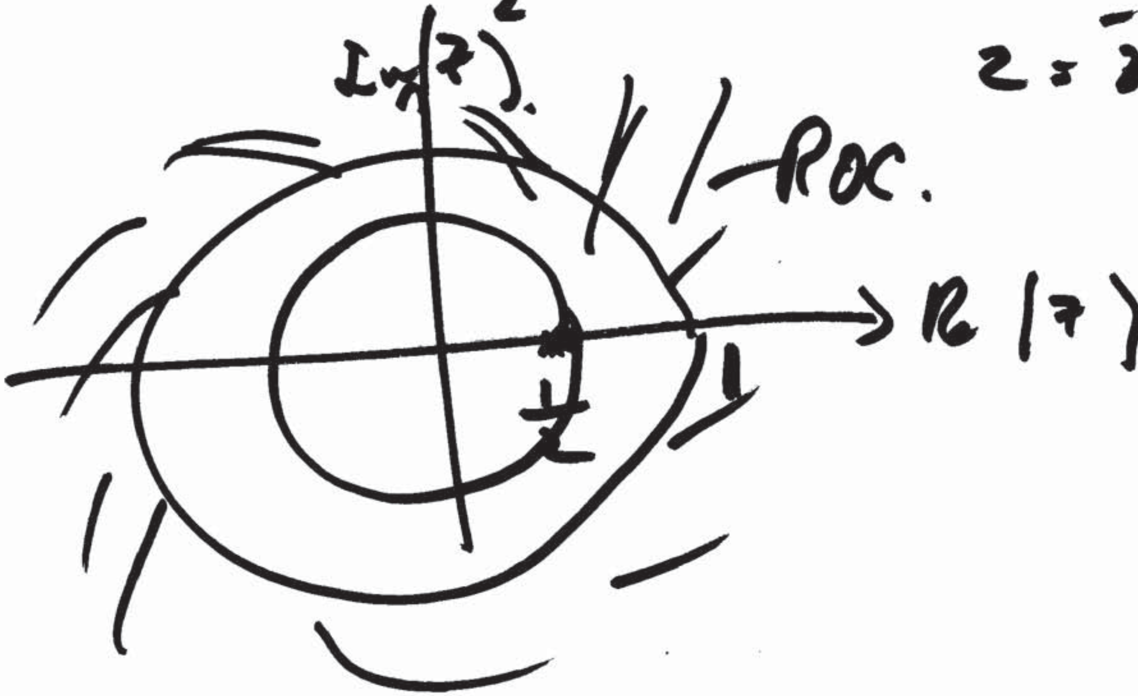
$$y(n) - \frac{1}{2}y(n-1) = x(n)$$

$$y(n) = x(n) + \frac{1}{2}y(n-1)$$



pole $1 - \frac{1}{2} z^{-1} = 0 \Longrightarrow 1 = \frac{1}{2} z^{-1}$

$2 = z^{-1} \Longrightarrow z = \frac{1}{2}$

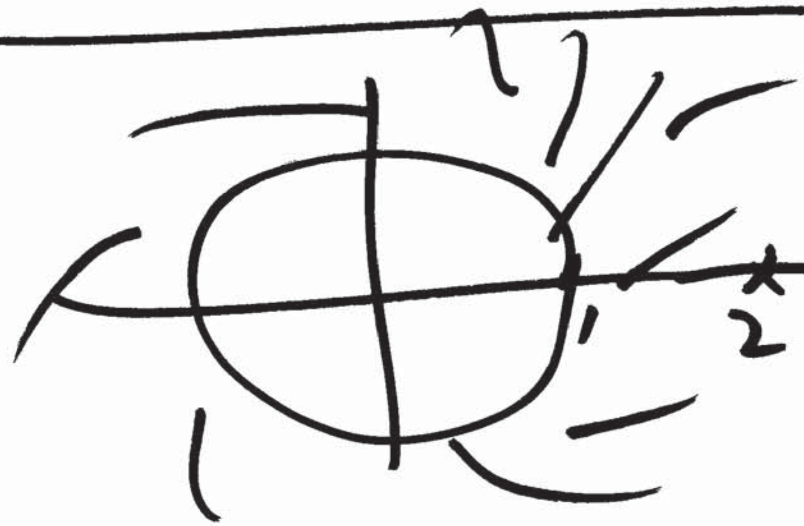


ROC.

$|z| > \frac{1}{2}$

$h(n) = \frac{1}{2}^{n} u(n)$

Unstable



$h(n) = 2^{n} u(n)$

$= 1 + 2 + 4 + 8 +$

$H(z) = \frac{1}{1 - 2z^{-1}}$

$H(z) = \dfrac{1}{1 + 5z^{-1} + 3z^{-2} + 9z^{-3} + 12z^{-4} + 18z^{-5}}$

2D polynomial can always Be factored.

→ Fundamental Thm of Algebra.

Any 1-D polynomial ~~coeff~~ of deg. n can be factored into n polynomials of deg. 1.

$= \dfrac{1}{\prod_{i=1}^{5} (1 - \alpha_i z^{-1})}$

Story in 2D

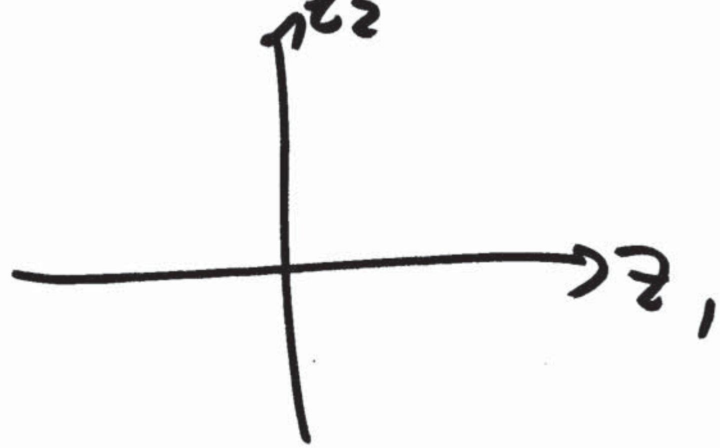
IIR: $H(z_1, z_2) = \dfrac{1}{P(z_1^{-1}, z_2^{-1})}$

$P(z_1^{-1}, z_2^{-1}) = (1 - z_1^{-1})(1 - z_2^{-1})$

Thm: Hayes 1980s:

Set of reducible 2D polynomials is of measure 0 in the set of all 2D polynomials.

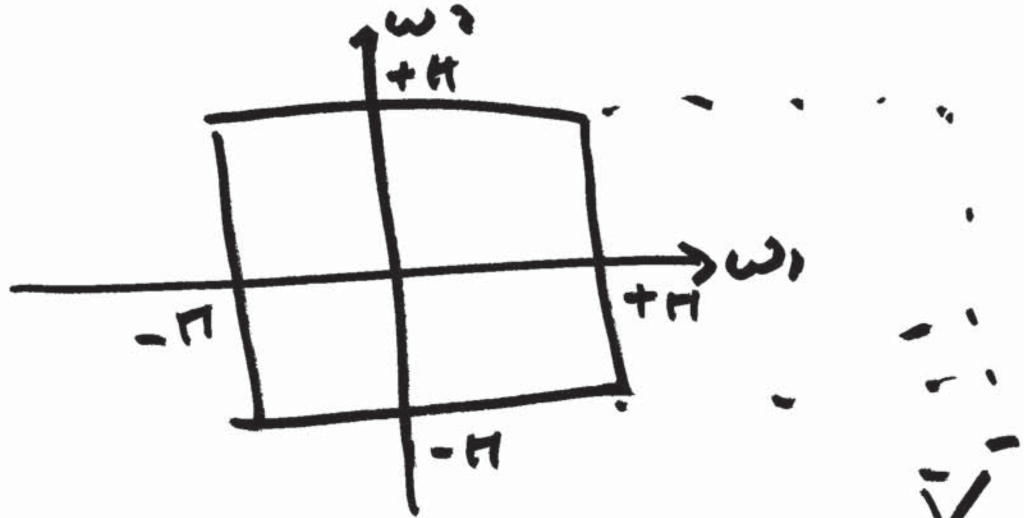
2D → unit bisurface



$\cancel{P(z_1^{-1}, z_2^{-1})}$

Do The pde surface cross The unit Bi-surface? $\Longrightarrow$ Some Thms covered later.

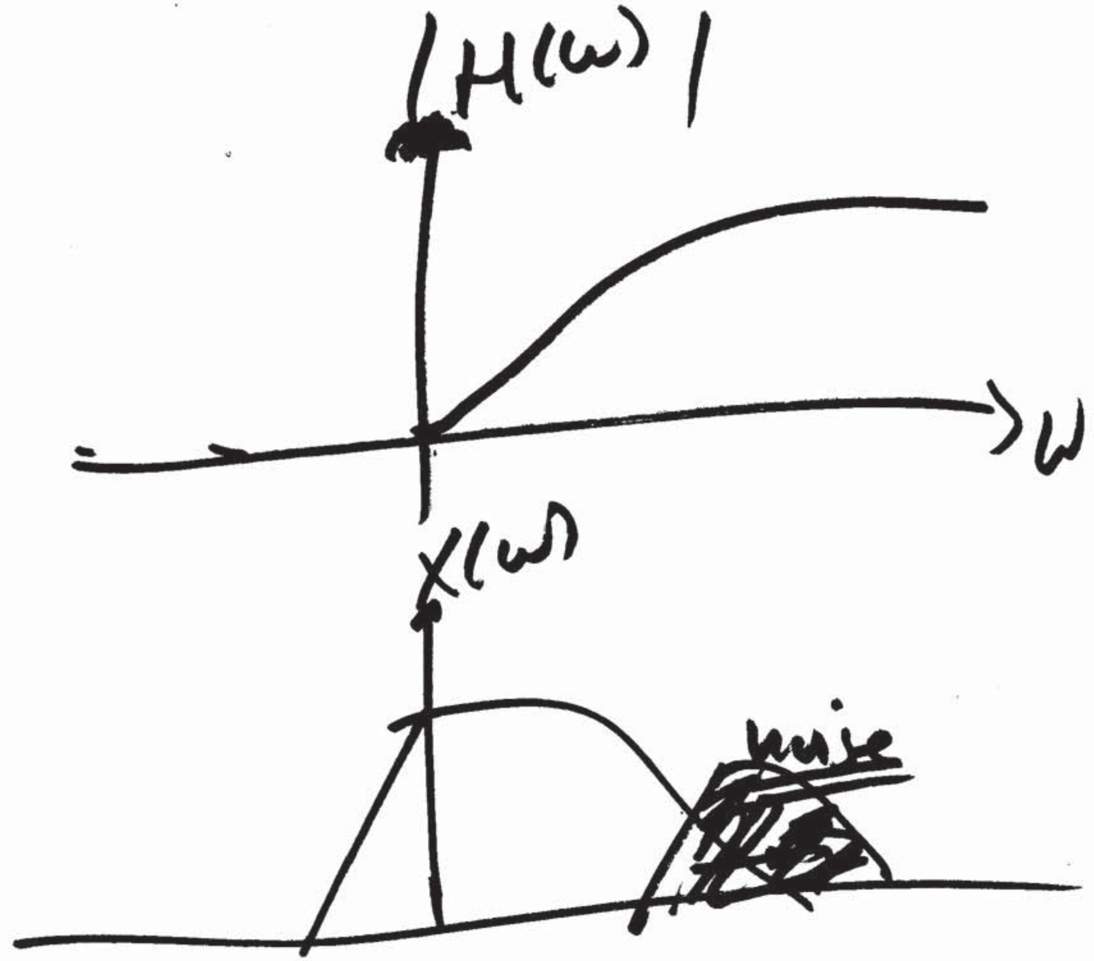
Proving stability in 2D is hard



$$x(n_1, n_2) \longrightarrow X(\omega_1, \omega_2)$$

discrete integer

Real variable.


|H(ω)|
ω
X(ω)
noise

Signal.