Fourier Transform (cont'd)

- More generally, Discrete Time Fourier Transform (DTFT) of a sequence $x(n_1, n_2)$ is given by:

$$X(\omega_1, \omega_2) = \sum_{n_1}\sum_{n_2} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \quad (14)$$

- Observations:

1. $X(\omega_1, \omega_2)$ is a complex function, has a real and imaginary part.
2. It is a continuous function of ω_1 and ω_2.
3. It is doubly periodic with period 2π in ω_1 and ω_2.
4. If $x(n_1, n_2)$ is separable, so is its Fourier Transform

$$\begin{aligned} x(n_1, n_2) &= x_1(n_1)x_2(n_2) \\ X(\omega_1, \omega_2) &= X_1(\omega_1)X_2(\omega_2) \end{aligned} \quad (15)$$

- Inverse Fourier Transform:

$$x(n_1, n_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2 \quad (16)$$

Properties of Fourier Transform

- linearity.

$$a x_1 \; + \; b x_2 \quad \leftrightarrow \quad a X_1 \; + \; b X_2$$

- Spatial Shift:

$$x(n_1 - m_1, n_2 - m_2) \quad \leftrightarrow \quad e^{-j\omega_1 m_1 - j\omega_2 m_2} X(\omega_1, \omega_2)$$

- Multiplication in time corresponds to convolution in frequency domain:

$$c(n_1, n_2) x(n_1, n_2) \leftrightarrow$$

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} X(\theta_1, \theta_2) C(\omega_1 - \theta_1, \omega_2 - \theta_2) d\theta_1 d\theta_2$$

- Differentiation in frequency domain corresponds to multiplication by j in the space domain:

$$-j n_1 x(n_1, n_2) \leftrightarrow \partial X(\omega_1, \omega_2) / \partial \omega_1$$

$$-j n_2 x(n_1, n_2) \leftrightarrow \partial X(\omega_1, \omega_2) / \partial \omega_2$$

Properties of Fourier Transform (cont'd)

- Transposition:

$$x(n_2, n_1) \quad \leftrightarrow \quad X(\omega_2, \omega_1)$$

- Reflection:

$$\begin{aligned} x(-n_1, n_2) &\quad \leftrightarrow \quad X(-\omega_1, \omega_2) \\ x(n_1, -n_2) &\quad \leftrightarrow \quad X(\omega_1, -\omega_2) \\ x(-n_1, -n_2) &\quad \leftrightarrow \quad X(-\omega_1, -\omega_2) \end{aligned}$$

- Complex Conjugate:

$$x^*(n_1, n_2) \quad \leftrightarrow \quad X^*(-\omega_1, -\omega_2)$$

- Real and Imaginary Parts:

$$Re[x] \leftrightarrow \frac{1}{2}[X(\omega_1, \omega_2) \ + \ X^*(-\omega_1, -\omega_2)]$$

$$Im[x] \leftrightarrow \frac{1}{2}[X(\omega_1, \omega_2) \ - \ X^*(-\omega_1, -\omega_2)]$$

$$\frac{1}{2}[x(n_1, n_2) \ + \ x^*(-n_1, -n_2)] \leftrightarrow Re[X(\omega_1, \omega_2)]$$

$$\frac{1}{2}[x(n_1, n_2) \ - \ x^*(-n_1, -n_2)] \leftrightarrow Im[X(\omega_1, \omega_2)]$$

Properties of Fourier Transform (cont'd)

- Parseval's theorem:

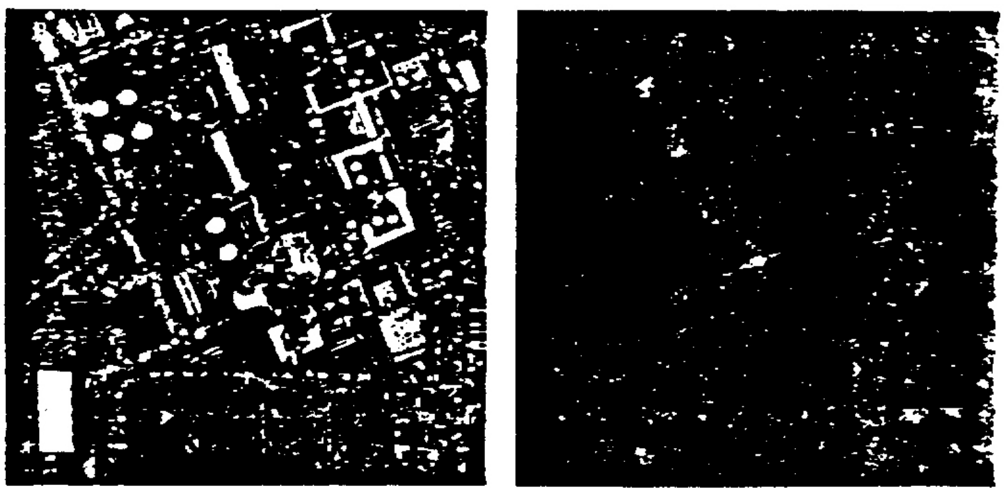
$$\sum_{n_1}\sum_{n_2} x(n_1, n_2) w^*(n_1, n_2) =$$

$$(\frac{1}{2\pi})^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) W^*(\omega_1, \omega_2) d\omega_1 d\omega_2$$

- Examples of Fourier Transform.

and along the ω_1 and ω_2 axes. One reason for the energy concentration near the origin is that images typically have large regions where the intensities change slowly. Furthermore, sharp discontinuities such as edges contribute to low-frequency as well as high-frequency components. The energy concentration along the ω_1 and ω_2 axes is in part due to a rectangular window used to obtain a finite-extent image. The rectangular window creates artificial sharp discontinuities at the four boundaries. Discontinuities at the top and bottom of the image contribute energy along the ω_2 axis and discontinuities at the two sides contribute energy along the ω_1 axis. Figure 1.33 illustrates this property. Figure 1.33(a) shows an original image of 512×512 pixels, and Figure 1.33(b) shows $|X(\omega_1, \omega_2)|^{1/4}$ of the image in Figure 1.33(a). The operation $(\cdot)^{1/4}$ has the effect of compressing large amplitudes while expanding small amplitudes, and therefore shows $|X(\omega_1, \omega_2)|$ more clearly for higher-frequency regions. In this particular example, energy concentration along approximately diagonal directions is also visible. This is because of the many sharp discontinuities in the image along approximately diagonal directions. This example shows that most of the energy is concentrated in a small region in the frequency plane.

Since most of the signal energy is concentrated in a small frequency region, an image can be reconstructed without significant loss of quality and intelligibility from a small fraction of the transform coefficients. Figure 1.34 shows images that were obtained by inverse Fourier transforming the Fourier transform of the image in Figure 1.33(a) after setting most of the Fourier transform coefficients to zero. The percentages of the Fourier transform coefficients that have been preserved in



(a) (b)

Figure 1.33 Example of the Fourier transform magnitude of an image. (a) Original image $x(n_1, n_2)$ of 512×512 pixels; (b) $|X(\omega_1, \omega_2)|^{1/4}$, scaled such that the smallest value maps to the darkest level and the largest value maps to the brightest level. The operation $(\cdot)^{1/4}$ has the effect of compressing large amplitudes while expanding small amplitudes, and therefore shows $|X(\omega_1, \omega_2)|$ more clearly for higher-frequency regions.

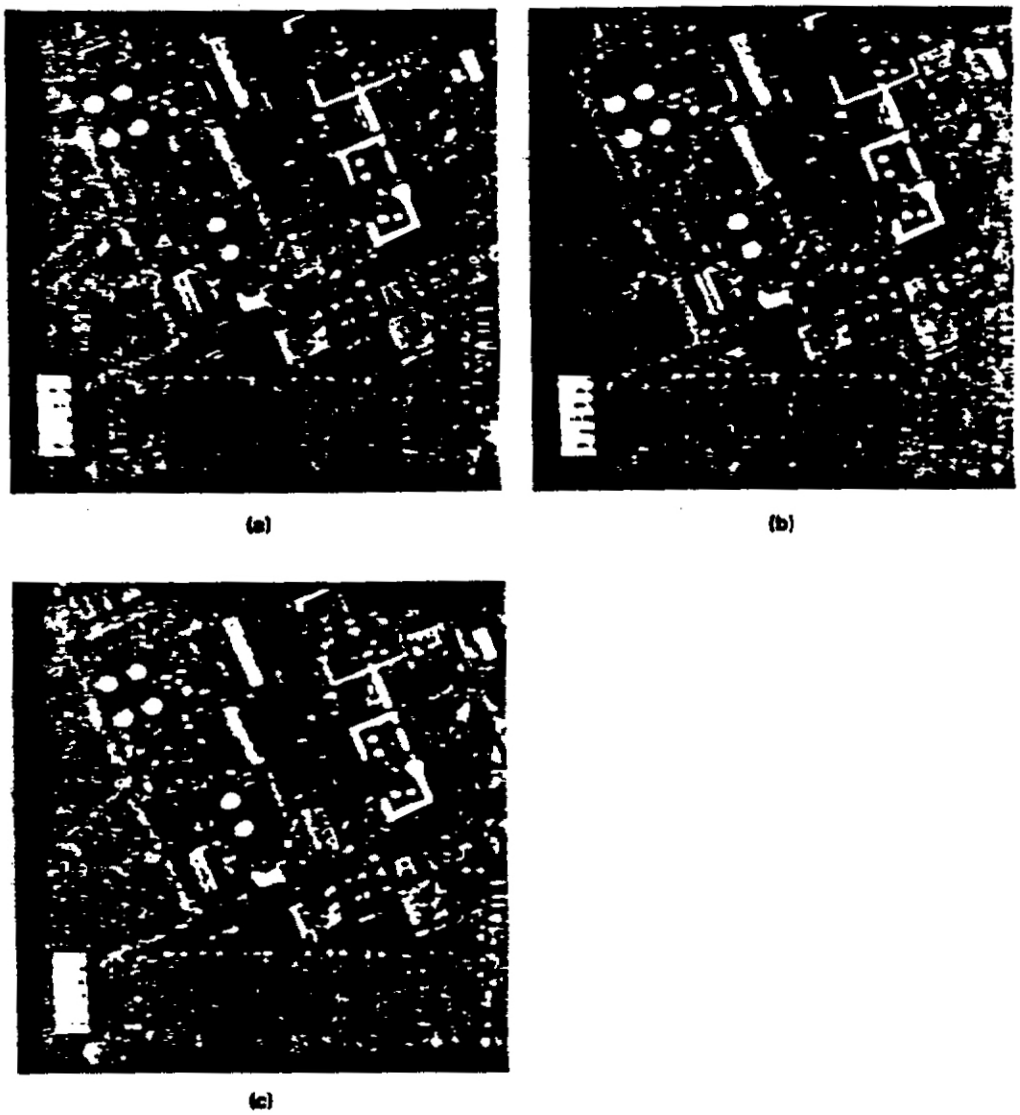


Figure 1.34 Illustration of energy concentration in the Fourier transform domain for a typical image. (a) Image obtained by preserving 12.4% of Fourier transform coefficients of the image in Figure 1.33(a). All other coefficients are set to 0. (b) Same as (a) with 10% of Fourier transform coefficients preserved; (c) same as (a) with 4.8% of Fourier transform coefficients preserved.