March 8, 2006

Mathematical Aspects/ Derivation of
Histogram Equalization

1. Consider continuous values of intensity.
   Rather than discretized.

2. $r$ is gray level of image to be equalized.

   $r$ normalized to $[0, 1]$

   $0 \rightarrow$ dark = Black

   $1 \rightarrow$ bright = White.

3. Goal: Design a Transformation:

   \[ T(r) = S, \quad 0 \leq r \leq 1, \]

   Assumptions about $T(r)$. 

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1
(a) $T(r)$ single valued and monotonically increasing
in the interval $0 \leq r \leq 1$

for inverse transform to exist

preserve increasing order from black to white in outputting.
(b) \( 0 \leq T(r) \leq 1 \) for \( 0 \leq r \leq 1 \)

\( \) output grey levels in the same range as input levels.

\[
T(r) = s \\
T^{-1}(s) = r
\]

\( r = \text{input intensity} \)  
\( s = \text{output intensity} \)

Result from prob. They:

If \( T^{-1}(s) \) satisfies condition (a) Then.

\[
P_s(s) = P_r(r) \left| \frac{dr}{ds} \right|
\]
Consider CDF as a Trapezium.

\[ S = T(r) = \int_0^r P_r(w) \, dw \]

\[ \frac{dS}{dr} = \frac{d}{dr} \left( \int_0^r P_r(w) \, dw \right) = P_r(r) \]

\[ P_s(s) = P_r(r) \left| \frac{1}{P_r(r)} \right| = 1 \quad 0 \leq s \leq 1 \]

**Words:** If \( T(r) \) is just a CDF or just the integral of input pdf \( (P_r(r)) \), then applying \( T(r) \) results in a image whose pdf \( (P_s(s)) \) is Uniform.
Discrete Case

\[ \Gamma_k = \text{discrete intensity value} \quad k = 0, \ldots, L-1 \]

\[ P_r(\alpha r_k) = \frac{n_k}{n} \quad k = 0, \ldots, L-1 \]

\[ n_k = \text{number of pixels that have intensity } r_k \]

\[ S_k = T(r_k) = \sum_{j=0}^{K} P_r(r_j) \left\langle \frac{k}{n} \right. \]

\[ S_k = \sum_{j=0}^{K} \frac{n_j}{n} \quad k = 0, \ldots, L-1 \]
Histogram Matching

\[ T(r) = S \]

Rather than \( P_3(s) \) uniform, we want \( P_3(s) \) to match a "desired" pdf given.

\[ r = \text{pixel value before matching} \]
\[ z = \text{pixel \"after\" matching} \]

Can compute \( P_z(z) \) from given image.

Know, given \( P_z(z) \)

Goal: what is the transformation \( r \rightarrow z \)?
Approach: \[ S = T(r) = \int_0^r P_r(w) \, dw \]

CDF of \( r \).

\[ v = G(z) = \int_0^z P_z(t) \, dt \]

CDF of \( z \).

Histogram matching

\[ G(z) = T(r) \]

\[ z = G^{-1} \{ T(r) \} \]

\[ T(z) \]

\[ T(r) \]

\[ T(0) \]

\[ T(\infty) \]

\[ 0 \rightarrow r_0 \rightarrow r_1 \rightarrow \infty \]

\[ v_0 \rightarrow \text{how to find } z_0 \]
Lookuptable

<table>
<thead>
<tr>
<th>$r_0$</th>
<th>$z_0$</th>
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<tbody>
<tr>
<td>$r_1$</td>
<td>$z_1$</td>
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$\rightarrow \text{Matched histogram of } P_\theta(z) \text{ desired}$