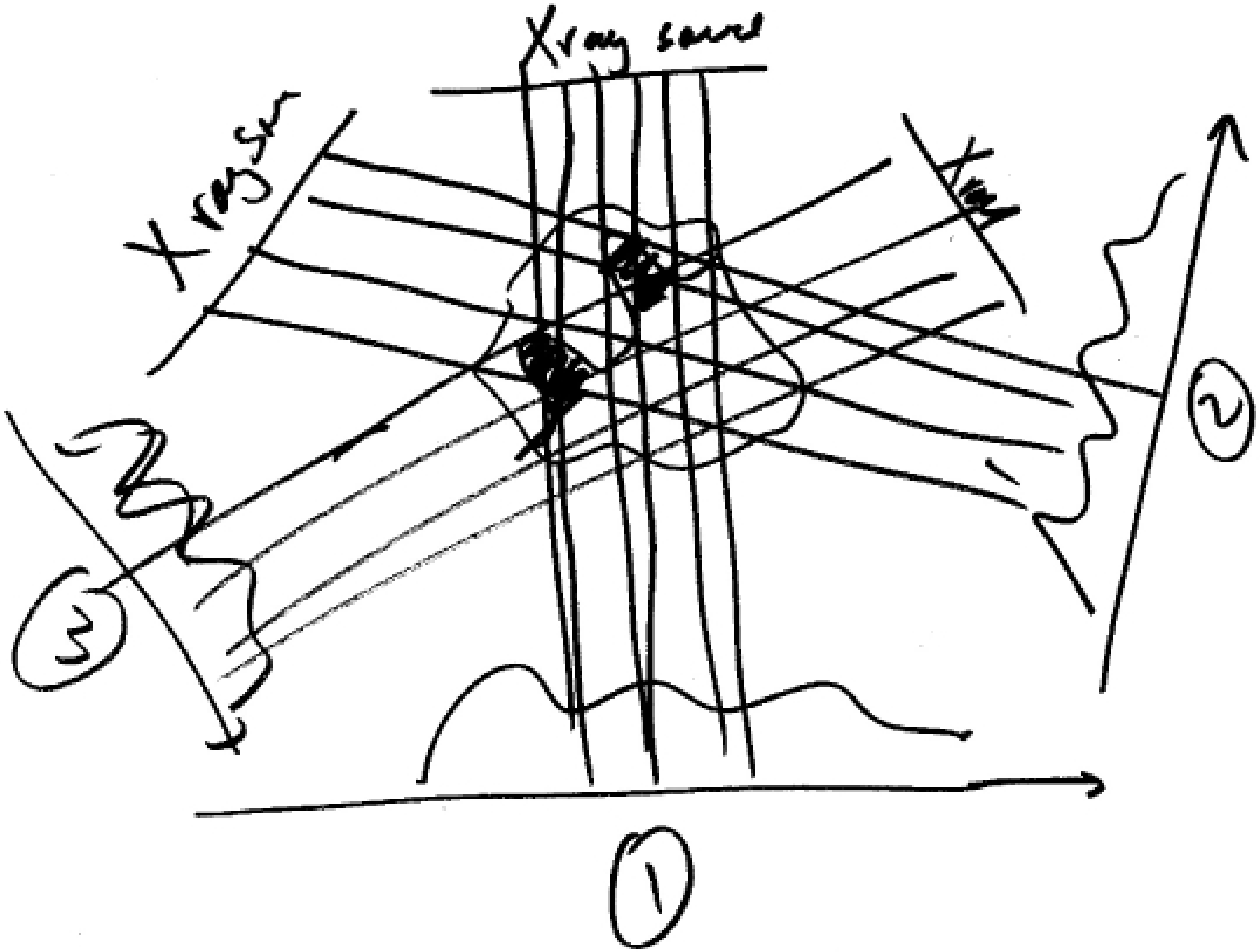
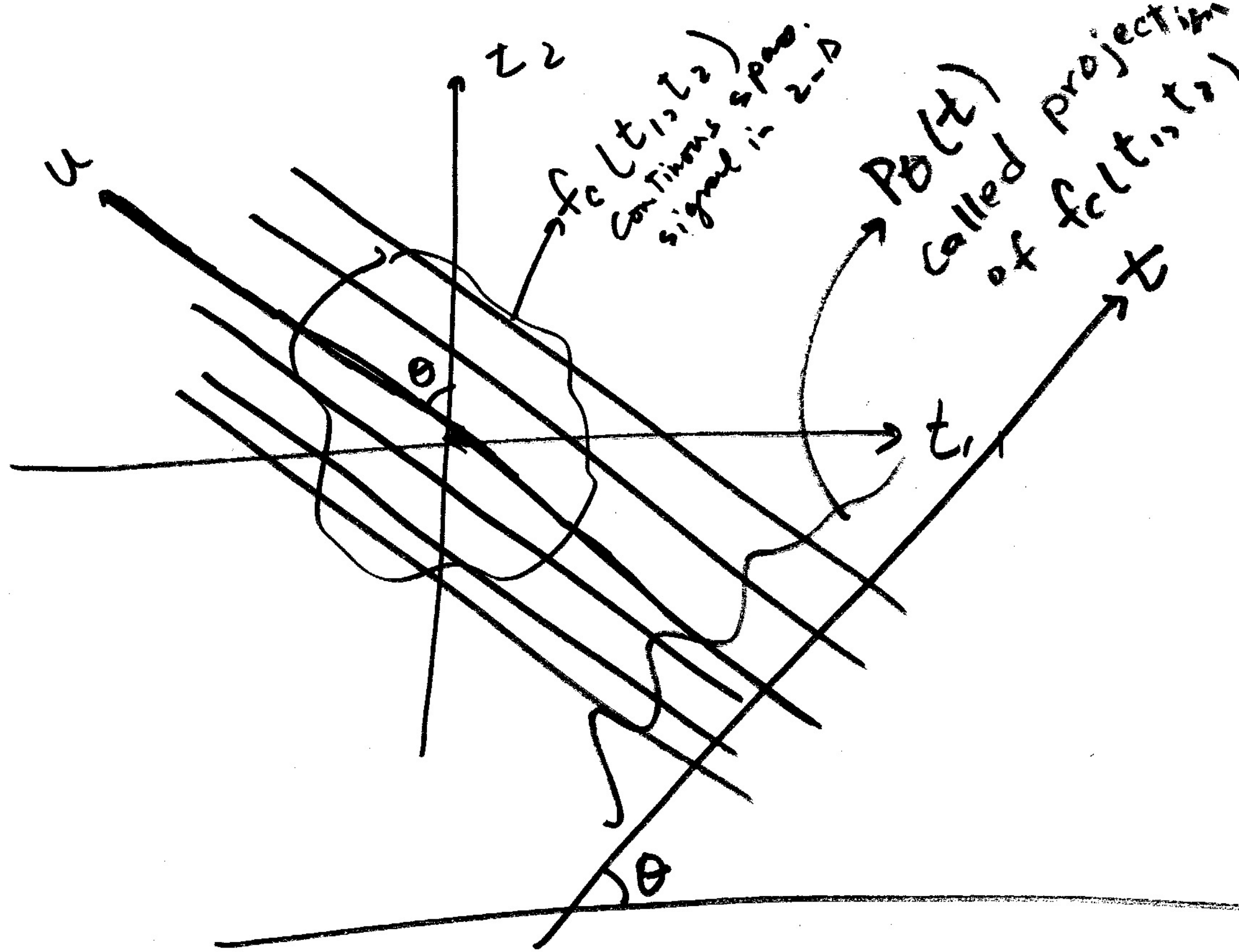
Jan. 30, 2008

Tomography



__Def__ Projection: is mathematical operation that is similar to the physical operation of taking an X-ray photograph with a collimated beam of radiation


t_2
u
$f_c(t_1, t_2)$
continuous space signal in 2-D
$P_\theta(t)$
called projection of $f_c(t_1, t_2)$
θ
t_1
t
θ

$$P_\theta(t) = \int_{u=-\infty}^{+\infty} f_c(t_1, t_2)\Big|_{\substack{t_1 = t\cos\theta - u\sin\theta \\ t_2 = t\sin\theta + u\cos\theta}} du$$

projection

<u>Goal</u>: Relate 2D F.T. of f_c to 1D. F.T. of P_θ

<u>Projection Slice Theorem</u>:

$$F_c(\Omega_1, \Omega_2) = \text{2.D. C.T.F.T.} \{ f_c(t_1, t_2) \}$$

$$F_c(\Omega_1, \Omega_2) = \int_{t_1=-\infty}^{+\infty} \int_{t_2=-\infty}^{+\infty} f_c(t_1, t_2)\, e^{-j\Omega_1 t_1} e^{-j\Omega_2 t_2}$$

$$f_c(t_1, t_2) = \frac{1}{4\pi^2} \iint F_c(\Omega_1, \Omega_2)\, e^{j\Omega_1 t_1} e^{j\Omega_2 t_2}\, d\Omega_1 d\Omega_2$$

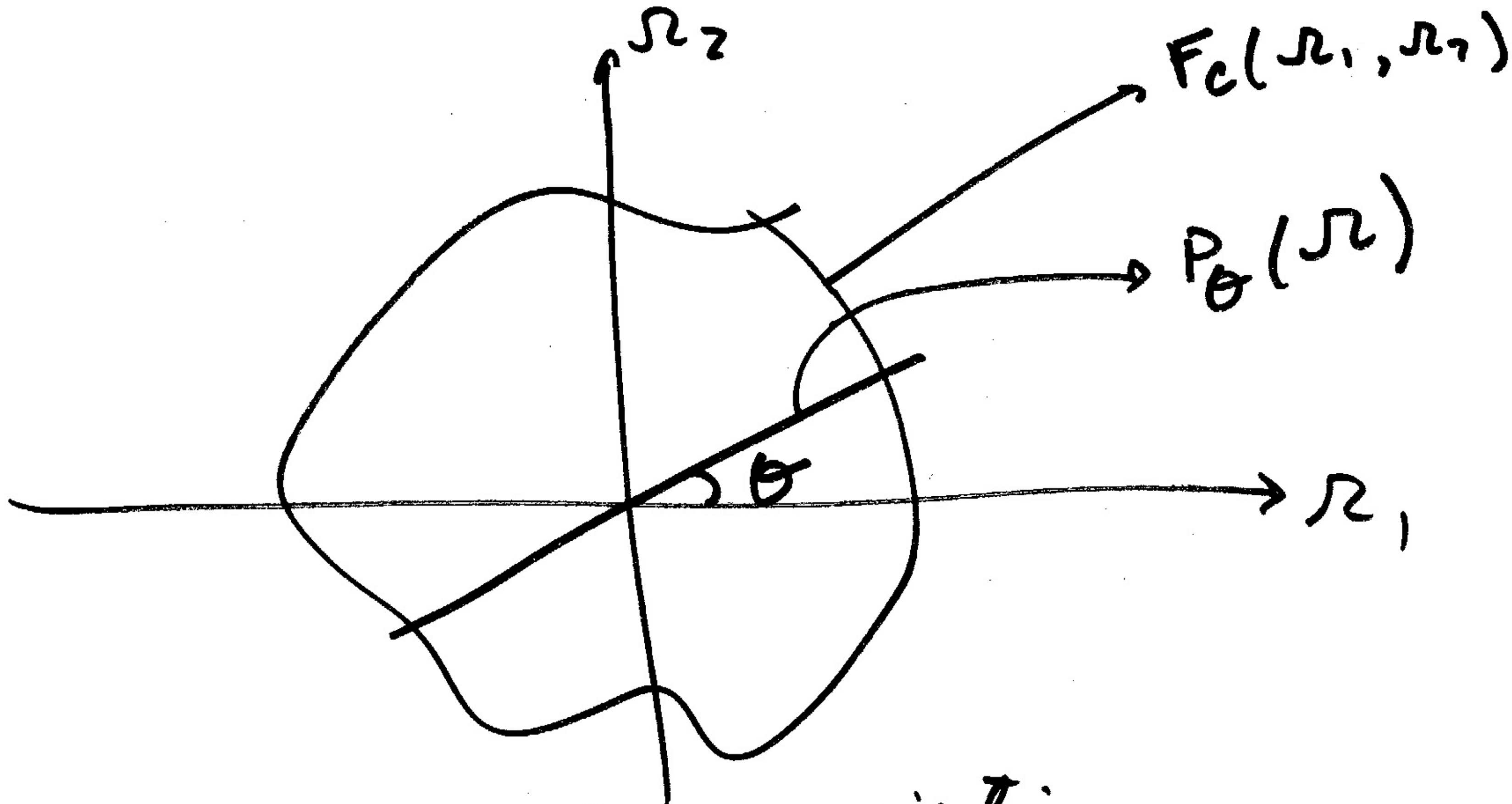
$$P_\theta(\Omega) = \text{1.D. C.T.F.T.} \{ p_\theta(t) \}$$

$$P_\theta(\Omega) = \int_{t=-\infty}^{+\infty} p_\theta(t)\, e^{-j\Omega t}\, dt$$

Projection Slice Theorem:

$$P_\theta(\Omega) = F_c(\Omega_1, \Omega_2) \Big|_{\substack{\Omega_1 = \Omega \cos\theta \\ \Omega_2 = \Omega \sin\theta}}$$

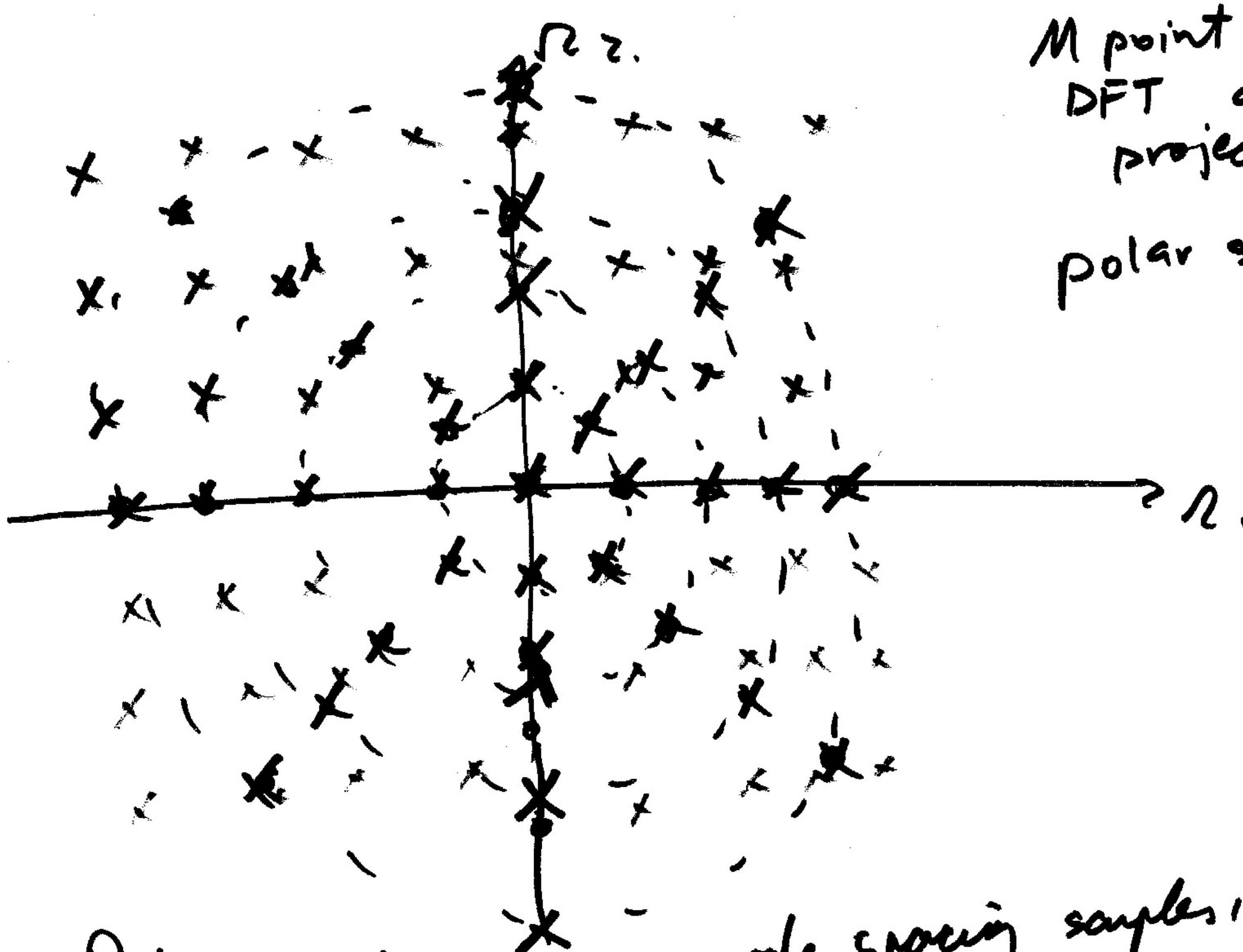
$$P_\theta(\Omega) = F_c(\Omega \cos\theta, \Omega \sin\theta)$$



By Taking multiple projection - you get 2D F.T. of the object

$\longrightarrow$ Inverse 2D. F.T

$\longrightarrow$ you get the object $f_c(t_1, t_2)$



M point DFT of projection.

polar sampling

Polar Sampling: Every angle spacing samples in Fourier domain are equal.

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To get blue cartesian samples from red polar sample, interpolate.

$\Rightarrow$ zeroth order $\longrightarrow$ nearest neighbor

1st order $\longrightarrow$ weighted sum of neighboring samples.

Concentric Squares: Distance between projection samples in F.D. varies a function of projection angle.

Reconstruction Strategies

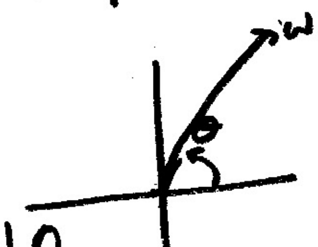
1. Simple
   - → nearest neighbor (zeroth order interp)
   - → 1st order Interpolation
2. Radon Inversion Formula
3. Iterative Technique

Radon Inversion Formula

$$f_c(t_1,t_2) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int F_c(\Omega_1,\Omega_2)\, e^{j\Omega_1 t_1}\, e^{j\Omega_2 t_2}\, d\Omega_1\, d\Omega_2$$

convert to polar coordinates $\Omega_1, \Omega_2 \longrightarrow \omega, \theta$

$$f_c(t_1,t_2) = \frac{1}{4\pi^2}\int_0^{\pi}\int_{-\infty}^{+\infty} \underbrace{F_c(\omega\cos\theta, \omega\sin\theta)}_{P_\theta(\omega)}\, e^{j\omega(t_1\cos\theta + t_2\sin\theta)}\, |\omega|\, d\omega\, d\theta$$



$$f_c(t_1,t_2) = \frac{1}{4\pi^2}\int_0^{\pi}\underbrace{\int_{-\infty}^{+\infty} P_\theta(\omega)\, e^{j\omega(t_1\cos\theta + t_2\sin\theta)}\, |\omega|\, d\omega}_{I}\, d\theta$$

$$I = \left\{ F^{-1}\left[P_\theta(\omega)\,|\omega| \right] \right\}_{t = t_1\cos\theta + t_2\sin\theta}$$

Define $G_\theta(\omega) = P_\theta(\omega)\,|\omega|$

$$F^{-1}\{G_\theta(\omega)\} = g_\theta(t)$$

$$I = \int_{-\infty}^{+\infty} G_\theta(\omega)\, e^{j\omega(t_1\cos\theta + t_2\sin\theta)}\, d\omega$$

$$I = \left[g_\theta(t)\right]_{t = t_1\cos\theta + t_2\sin\theta}$$

$$I = g_\theta(t_1\cos\theta + t_2\sin\theta)$$

$$\boxed{f_c(t_1,t_2) = \frac{1}{4\pi^2}\int_0^\pi g_\theta(t_1\cos\theta + t_2\sin\theta)\, d\theta} \quad *$$

Recall $G_\theta(\omega) = P_\theta(\omega)\,|\omega|$

$$g_\theta(t) = K(t) * P_\theta(t)$$

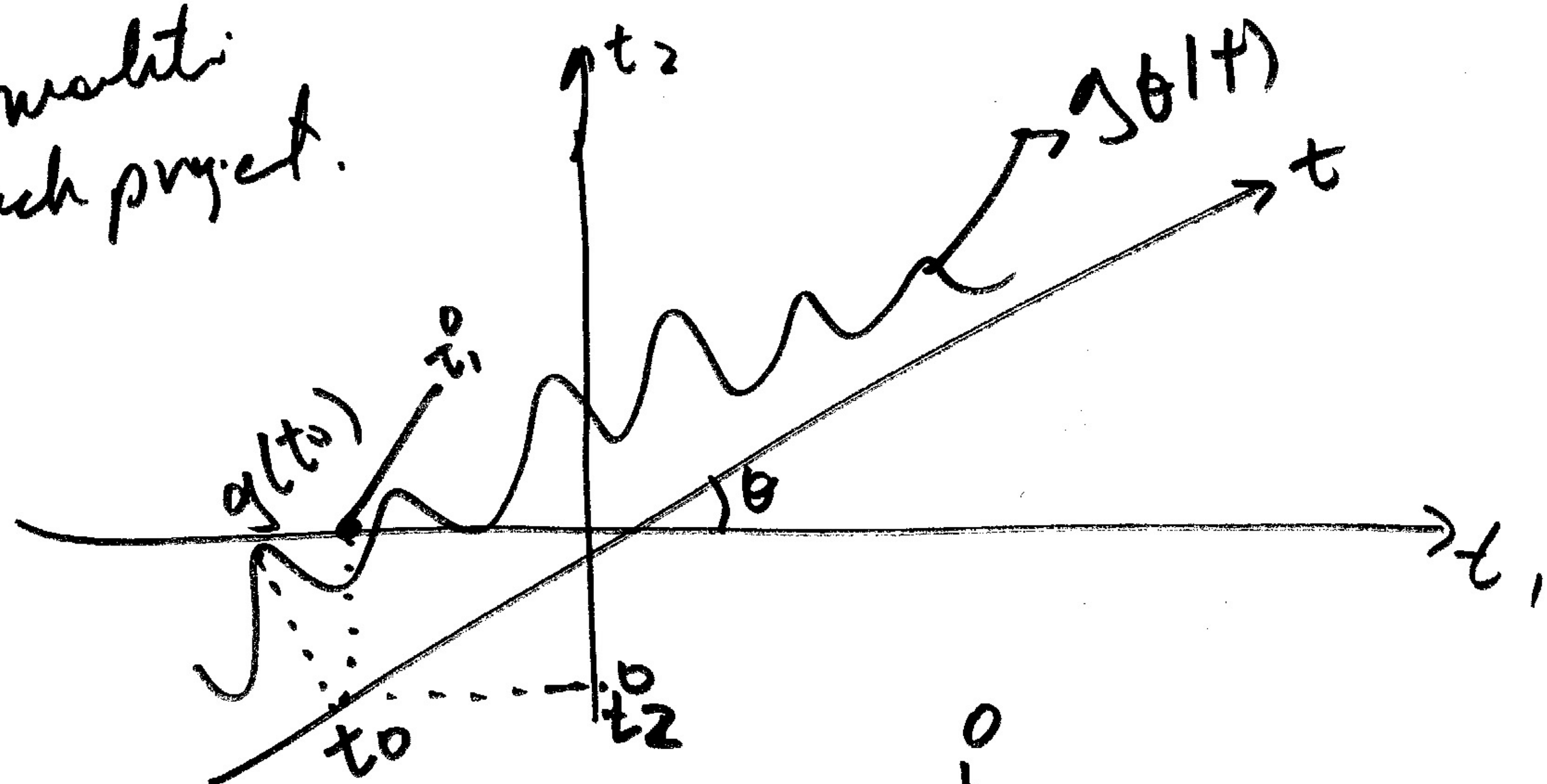
where $K(t) = \mathcal{F}^{-1}\{|\omega|\}$

$$\to g_\theta(t) = \frac{d}{dt}\int_{-\infty}^{+\infty} \frac{P_\theta(\tau)}{t-\tau}\,d\tau$$

Steps:

1. $P_\theta(t)$
2. convolve with $K(t)$ to get $g_\theta(t)$
3. plug $g_\theta(t)$ into $*$

Convoluti
Back project.



at $t = t_0$

$t_0 \to t_1^0,\ t_2^0$

$\cancel{f_c(t_1^0, t_2^0) = g(t_0)}$

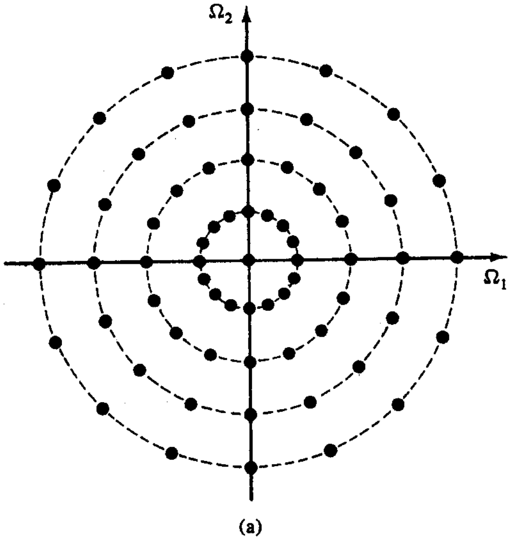
Iterative Recon

$$f_c^{K}(t_1, t_2) = f_c^{K-1}(t_1, t_2) + \sum_{i=1}^{N} \lambda_i \left[P_{\theta_i}(t_1 \cos\theta + t_2 \sin\theta) - D_i\{ f_c^{K-1}(t_1, t_2) \} \right]$$

actual projection observed

determines convergence rate

projection operator applied to Kth iteration reconstruction



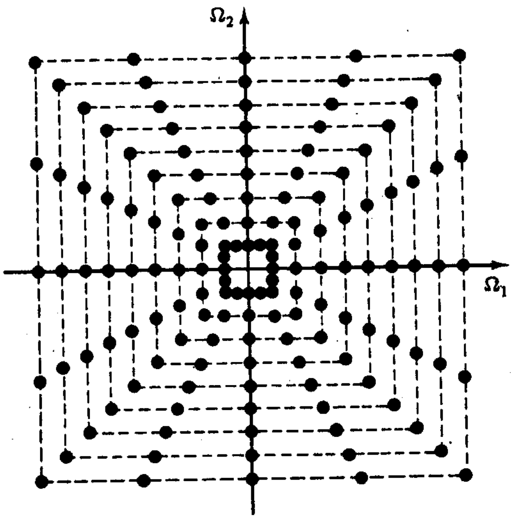


Figure 7.13 (a) Polar raster of samples in the Fourier domain, obtained by sampling all projections at the same sampling rate. (b) Concentric squares raster, obtained by varying the sampling rate with the angle of the projection. (Courtesy of Russell M. Mersereau, *Proc. IEEE*, © 1974 IEEE.)

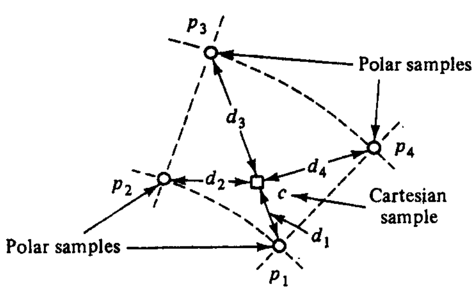
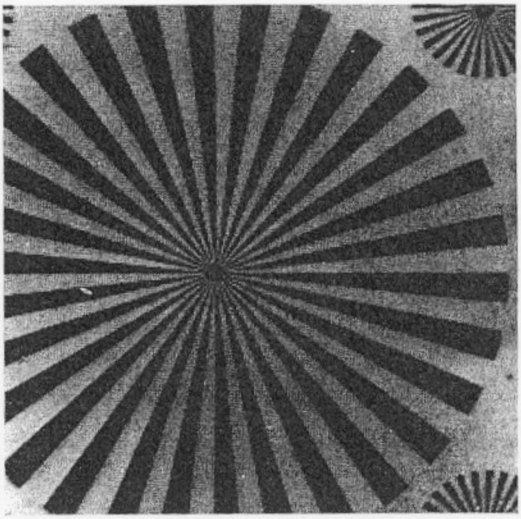
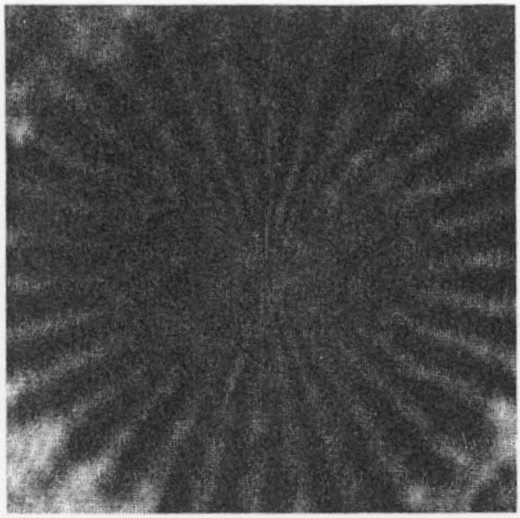


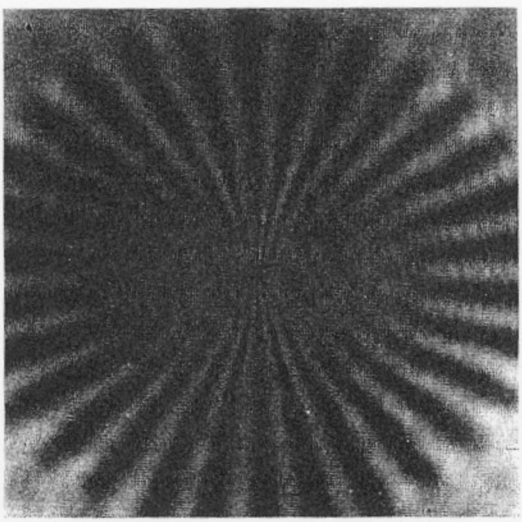
Figure 7.14 Parameters for the definition of zeroth-order and linear interpolation. (Courtesy of Russell M. Mersereau, *Proc. IEEE*, © 1974 IEEE.)



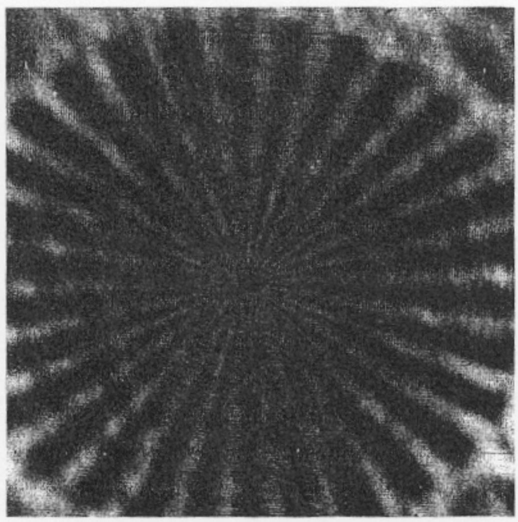
(a)



(b)



(c)



(d)

Figure 7.15 Reconstructions of the original image shown in (a) made from 64 equiangular projections using various interpolation algorithms. (b) Zeroth-order interpolation, polar raster. (c) Linear interpolation, polar raster. (d) Linear interpolation, concentric squares raster. (Courtesy of Russell M. Mersereau, *Proc. IEEE*, © 1974 IEEE.)

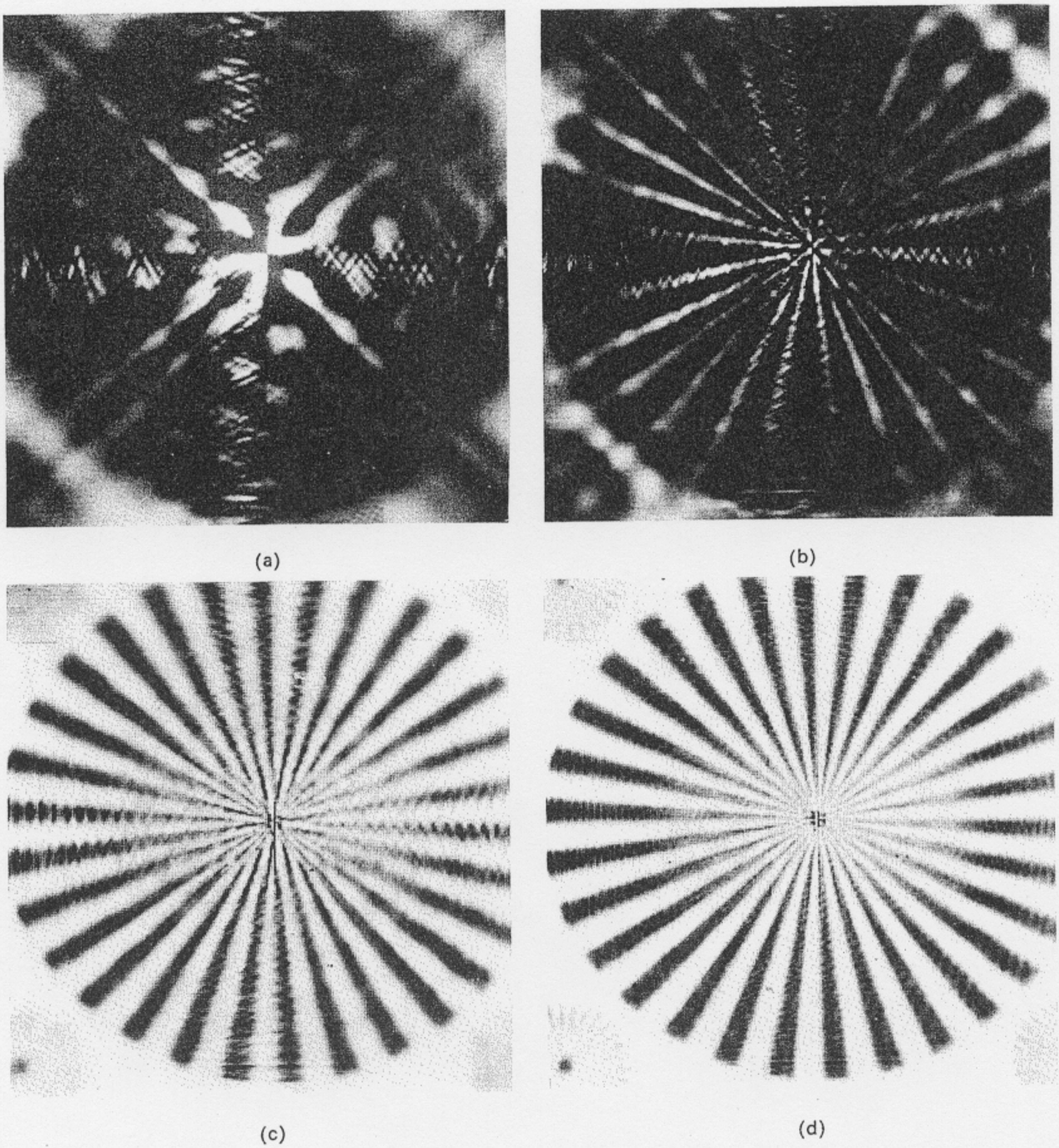
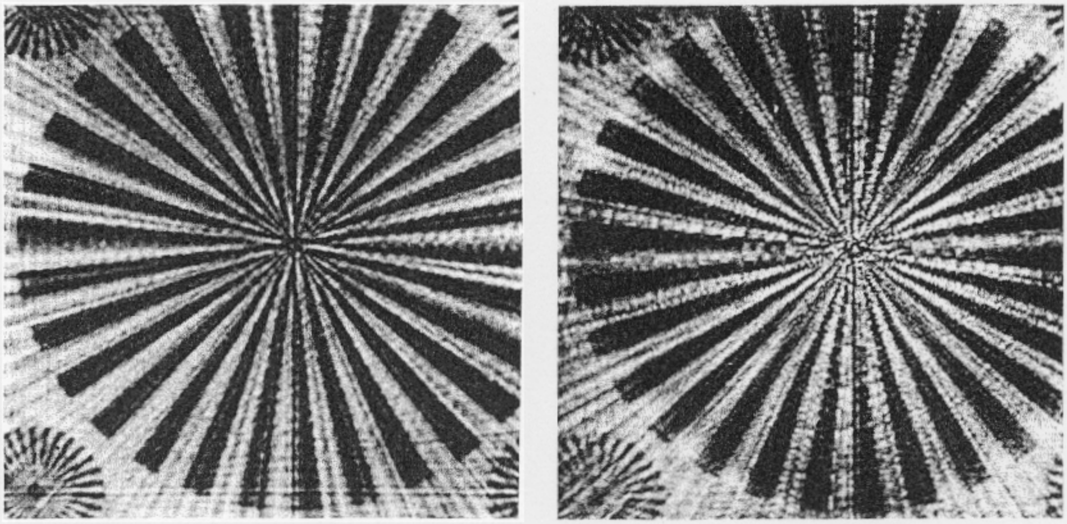


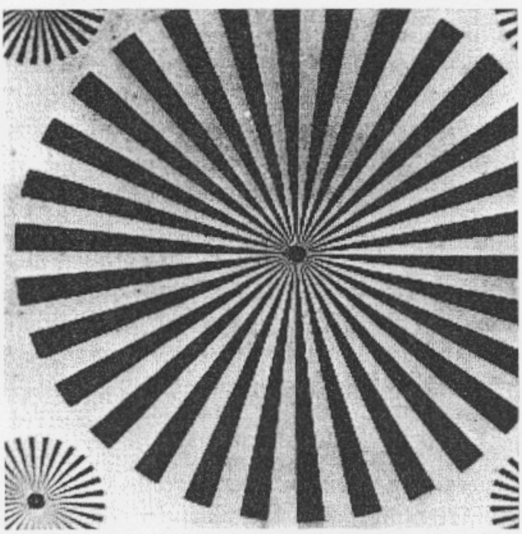
Figure 7.16 Reconstructions made using linear interpolation from a concentric squares raster using: (a) 16 projections; (b) 32 projections; (c) 64 projections; (d) 128 projections. (Courtesy of Russell M. Mersereau, *Proc. IEEE*, © 1974 IEEE.)

filter gain increases with increasing frequency, high-frequency noise will be amplified. Thus to minimize the deterioration that can result from such noise, the filter $k(t)$ is typically chosen to have an approximately linear response out to some cutoff frequency beyond which the response goes to zero. The exact shape of the frequency response is also governed by computational convenience [20, 21].

Some reconstructions obtained using this algorithm are shown in Figures 7.17 and 7.18. The resolution here is noticeably better than for the reconstructions

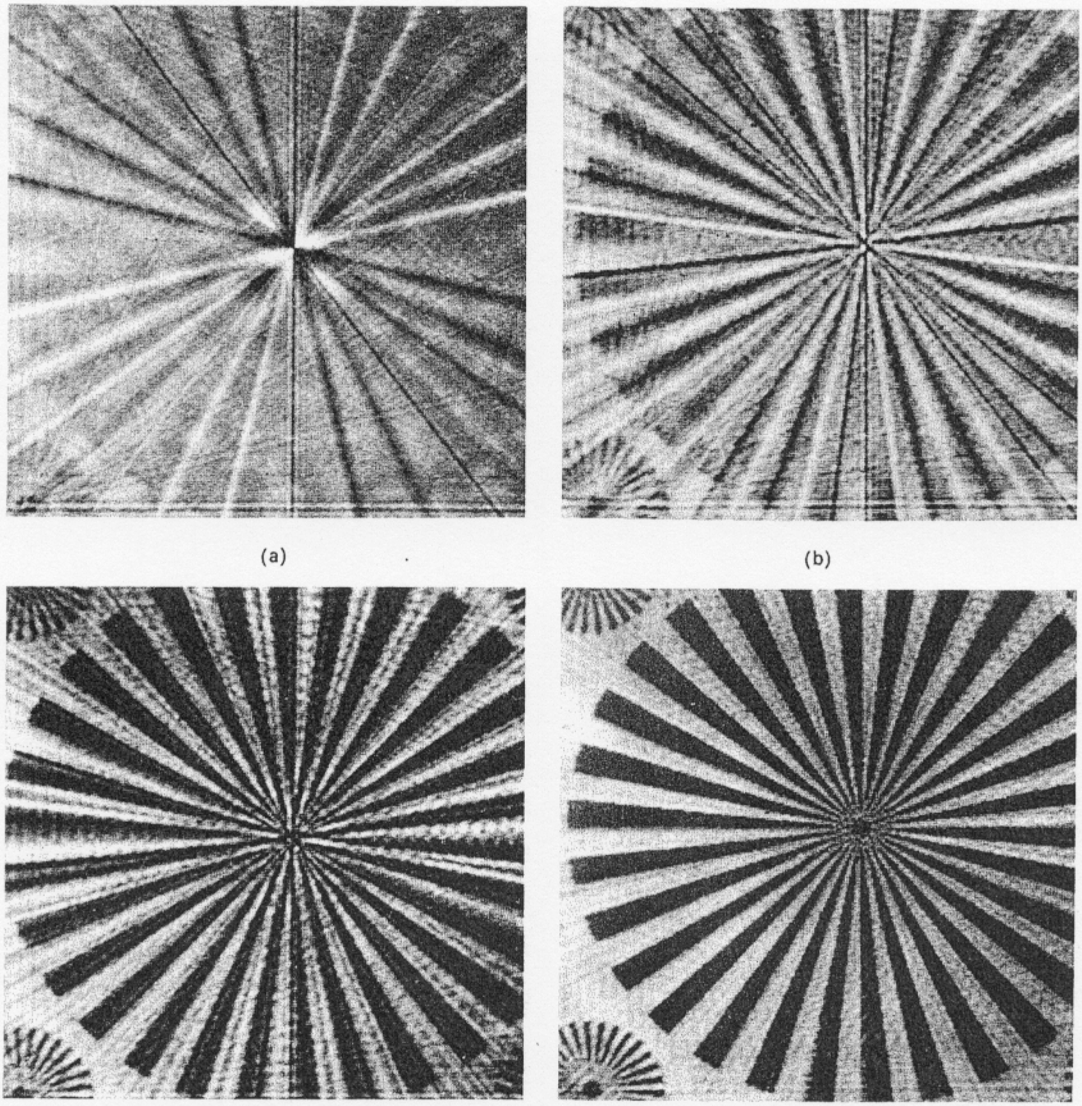


(a) (b)



(c)

Figure 7.17 Reconstruction of the original image shown below from 64 equiangular projections using the convolution/back-projection method. (a) Concentric squares projections. (b) Polar projections. (c) Original. (Courtesy of Russell M. Mersereau, *Proc. IEEE*, © 1974 IEEE.)



(a) (b) (c) (d)

Figure 7.18 Reconstructions made using the convolution/back-projection method applied to concentric squares projections. (a) 16 projections. (b) 32 projections. (c) 64 projections. (d) 128 projections. (Courtesy of Russell M. Mersereau, *Proc. IEEE,* © 1974 IEEE.)