Reconstruction of 2.D Signals
from Partial Fourier Information

Motivation:
- electron microscopy
- optical astronomy
- crytology

3 Problems:
1. FTM = magnitude of Fourier Trafo
2. Recons from Phase Fourier Trafo
3. Recons from Level
1-D case:

- \( X(n) \): discrete time 1-D.

\[ n = 0, \ldots, N \]

\[ N+1 \text{ non-zero point} \]

D.T.F.T \[ \{x(n)\} = X(w) = \sum_{n=0}^{N} x(n) e^{-jwn} \]

If we know \( |X(w)| \) \( \rightarrow \) Can we get \( x(n) \) ?

If I know \( |X(w)| \) at \( N+1 \) pts \( \rightarrow \) Cannot reconstruct signal.
\[
\begin{bmatrix}
0.5w & 0.5 & 0.5
\end{bmatrix}
\begin{bmatrix}
x(0) \\
x(1) \\
x(n)
\end{bmatrix}
= 
\begin{bmatrix}
1/X(\omega_0) \\
1/X(\omega_1) \\
1/X(\omega_n)
\end{bmatrix}
\]

\[
\sum_{n=0}^{N} x(n) y^n = X(\omega)
\]

Sample a 2-D poly edge at arbitrary point, result in unique recon of polynomial.
Q: Can we uniquely reconstruct if we oversampled F.D.?

A: For 1D signal → No ambiguity
    MD signal → Yes
auto-correlation fn of $x(n)$.

$$r(n) = \sum_{l} x(l) x^*(l+n)$$

$$R(w) = D.T.F.T \{ r(n) \} = \sum r(n) e^{-jwn} = |X(w)|^2 \Rightarrow$$

$$\Rightarrow F^{-1} \{ |X(w)|^2 \} = r(n)$$

Q: Can we obtain $x(n)$ from

$$r(n) = \sum_{n=-N}^{N} r(n) z^{-n}$$

Z.T. $\{ r(n) \}$: $R(z) = X(z) X^* \left( \frac{1}{z^*} \right) = \sum_{n=-N}^{N} r(n) z^{-n}$

Associated polynomial to $R(z) \equiv P_r(z) = \sum_{n=0}^{2N} r[N-n] z^n$

Since $P_r(z)$ is symmetric, if $z_0$ is a zero of $P_r(z)$, so is $z_0^{-1}$.
Can factor $P_r(\zeta)$

$P_r(\zeta) = A \prod_{i=1}^{N} (\zeta - \zeta_i) \prod_{i=1}^{N} (\zeta^* - \zeta_1^*)$

Def mirror of a polynomial.

Associated with any polynomial, there is a mirror polynomial consisting of coefficients in reverse order and conjugated.

If $P(\zeta) = \sum_{n=0}^{N} p_n \zeta^n$ $\longrightarrow$ mirror $\tilde{P}(\zeta) = \sum_{n=0}^{N} p_{N-n} \zeta^n$
- Assume $y(n) \rightarrow$ auto correlation $r(n)$

$$P_r(z) = P_y(z) \hat{r}_y(z)$$

$2^N$ ways in order to generate $P_y(z)$.

$$P_y(z) = \prod_{i=1}^{N} \prod_{j \neq i}^{\infty} (z - z_i) (z - z_j^*)$$

ie $I$ $j \notin I$

$I$ any subset of $[1, \ldots, N]$
2-D case F7M

Assume $|X(w_1, w_2)|^2$

$x(n_1, n) : N \times N$

⇒ in 2-D, unlike 1-D, most 2D polynomials are irreducible ⇒ non factorable.
1982, M. Hayen: If \( x(n_1, n_2) \) has irreducible associated polynomials, then all other \( y(n) \) which have some F.T.M are equivalent to \( x \). Equivalent means:

\[
y(n_1, n_2) = x(n_1, n_2) \quad \text{if} \quad y(n_1, n_2) = e^{j\theta} x(n_1, n_2)
\]

\[
y(n_1, n_2) = e^{j\theta} x(k_1 + n_1, k_2 + n_2)
\]

Observation 1: \( x(n_1, n_2) \rightarrow |X(w_1, w_2)|^2 \)

\[
e^{j\theta} x(n_1, n_2) \rightarrow |X(w_1, w_2)|^2
\]

Observation 2: \( x(n_1, n_2) \rightarrow |X(w, w_2)|^2 \)

Shift by \( k_1, k_2 \) in space domain \( \rightarrow \) same F.T.M.
Observation 3: Rotation factor

\[ x(n_1, n_2) \rightarrow |X(w_1, w_2)|^2 \]

\[ y(n_1, n_2) = x(N-n_1, N-n_2) \]

also has same FTM.

1. Assume signal \( x(n_1, n_2) \) is real

   \[ e^{j\theta} \]

   \[ \theta = 0 \text{ or } \theta = \pi \]

2. Assume signal is positive \( \Rightarrow \theta = 0 \)
2. Assumption: extent of $x(n_1,n_2)$ is known.

3. Z.T. of signal $x \rightarrow$ irreducible.

Hayes: Nail signal to factor of 2.

Either $x(n_1,n_2)$ or $x(N-n_1,N-n_2)$

Same F.T.M.

1. Assume $x$ real, written and positive.

Know extent.

F.T.M. can be used to recover either $x$ or its reflection "uniquely"
Proposed Iterative Alg

Assume known 4N x 4N samples of $x(w, wn)$

4N x 4N

Initial guess $N x N$

DFT $4N x 4N$

Resulting $4N x 4N$

Set extra coeff to 0

N x N

Keep phase

Replace Mag with what is given

I DFT $4N x 4N$

STOP

Yes

No

From step 0

O

O

O

O

O

O

O

O

O