Filter Specs

Low pass filter

$H(\omega)$

1

1 - $\delta_p$

$\delta_s$

$\delta_p = $ passband ripple
$\delta_s = $ stopband ripple

0.01
0.001

$\omega_p$
$\omega_s$

passband
stopband

transition band
2D specs

low pass filter in 2D

Stop band

Pass band

$|H(w_1, w_2)| < 1$

$|H(w_1, w_2)| < \delta_2$

$(w_1, w_2)$ outside $C_S$

$(w_1, w_2)$ inside $C_P$

$1 - \delta_P < |H(w_1, w_2)| < 1$
3 ways of designing FIR Filters

1. Window: \(1D \rightarrow 2D\)
2. Frequency sampling: \(1D \rightarrow 2D\)
3. Design by transformation: McClellan

Window

1. Start with ideal freq. response \(H_0(w)\)
2. Take inverse DFT \(h(n) \rightarrow \text{sinc. filter} \rightarrow \text{IIR}\)
3. \(h(n) = W(n) \cdot h_0(n)\)
\[ h(n) = W(n) \ast H_d(n) \]

\[ \left| H_d(\omega) \right| \]

\[ W(n) \ast H_d(n) \]

\[ \text{Length of } W(n) \]

\[ \text{shape } H(\omega) \]
\[ H(\omega) = |H(\omega)| e^{j\phi(\omega)} \]

Zero phase \( \phi(\omega) \) is linear in \( \omega \) →

Linear phase \( \phi(\omega) \) is zero
If $h$ is zero phase $Z \rightarrow h(z)$ is zero phase

If $W = 0$ then $W = 0$
\[ w_a(t) = \begin{cases} 
I_0 \left( \frac{\alpha \sqrt{1 - \left( \frac{t}{\Delta} \right)^2}}{I_0(x)} \right) & \text{if } t \leq \Delta \\
0 & \text{otherwise} 
\end{cases} \]

duration \leq 2\Delta

\[ \Delta = \text{Kurto } \text{Tractor off mainlobe width + } \text{sidelobe height} \]

\[ I_0(x) = \text{modified 2nd order Bessel fn} \]

\[ I_0(x) = \sum_{i=0}^{\infty} \frac{x^i}{2^i (i!)^2} \]

\[ 2^i \times i! \]
Continued time $2^b$ window.

$$W_a(t_1, t_2) = \begin{cases} \text{wa}(t) & \text{for discrete} \\ \text{(n,m)(ER)} & \text{otherwise} \end{cases}$$

Sample: $10$

$$w(a, n_1) = s_{c(t_1, t_2)}$$

$\theta$
\[ W(n_1, n_2) = W_a(n_1) W_a(n_2) \]

Sampled version of \( W_a(t) \).

\[ W(w_1, w_2) = W_a(w_1) W_b(w_2) \]
Another way 2D Windoors

Rotate \( \text{wa}(t) \) to get \( \text{wa}(t_1, t_2) \)

Sample \( \text{wa}(t_1, t_2) \) to get \( W(u_1, u_2) \)

\[
\text{wa}(t_1, t_2) = \text{wa}(t) \quad | \quad t = \sqrt{t_1^2 + t_2^2} \\
\text{wa}(t_1, t_2)
\]
\[ W(n_1, n_2) = \begin{cases} \text{Circular symmetric window} & \text{otherwise} \\ \begin{cases} W(t_1, t_2) \mid t_1 = n_1 \\ t_2 = n_2 \end{cases} & 0 \end{cases} \]

Show pictures from J. Lim

4.5 \rightarrow 4.9
Freq. Sampling Theory

Sample 2D Fourier space on a MxM equally spaced points.

\[ t \rightarrow u, \quad \text{IDFT} \rightarrow \text{MxM FIR filter} \]

Example: 4.10 \rightarrow 4.11
2D Filter Design using Transforma-

\[ H(u, v) = \begin{bmatrix} H_1(u) & H_2(u) \\ H_3(u) & H_4(u) \end{bmatrix} \]

1. How to design \( H_1(u) \) and \( H_2(u) \)?
2. How to design \( H_3(u) \) and \( H_4(u) \)?

- Design of \( H_1(u) \) and \( H_4(u) \):
  \[ H_1(u) = 20 \cos^2 \left( \frac{\pi u}{2} \right) \]
  \[ H_4(u) = 20 \cos^2 \left( \frac{\pi v}{2} \right) \]

- Design of \( H_2(u) \) and \( H_3(u) \):
  \[ H_2(u) = \frac{1}{2} \left( 1 + \cos \left( \frac{\pi u}{2} \right) \right) \]
  \[ H_3(u) = \frac{1}{2} \left( 1 + \cos \left( \frac{\pi v}{2} \right) \right) \]
\[ H(\omega) = H^*(\omega) \quad \Rightarrow \quad h(n) = h^*(-n) \]

Assume \( h(n) \) has real coeff.

\[ \Rightarrow h(n) = h(-n) \]

Zero phase in 1D \[ \quad \Rightarrow \quad h(n) = h(-n) \]

Assume \( H(\omega) \) is zero phase

\[ h(n) = h(-n) \]

\[ h(n) \text{ has } 2N+1 \text{ sample.} \]

\[
H(\omega) = \sum_{n=-N}^{N} h(n) e^{-j\omega n}
\]

\[ \text{Zeropane } L = h(0) + \sum_{n=1}^{N} 2h(n) \cos(\omega n) \]
\[ H(\omega) = \sum_{n=0}^{N} a(n) \cos(n \omega) \]

\[ H(\omega) = \sum_{n=0}^{N} b(n) (\cos(\omega))^n \]  

\[
\begin{align*}
\cos(2\omega) &= \cos(\omega + \omega) = 2 \cos^2 \omega - 1 \\
\cos(3\omega) &= \cos(\omega + 2\omega) = 2 \cos^2 \omega - \cos \omega \\
&= 2[\cos^2 \omega - \frac{(1+\cos^2 \omega)\cos \omega}{2}] 
\end{align*}
\]

\[ H(\omega_1,\omega_2) = \left[ H(\omega) \right]_{\cos \omega = T(\omega_1,\omega_2)} \\
H(\omega_1,\omega_2) = \sum_{n=0}^{N} b(n) \left[ T(\omega_1,\omega_2) \right]^n \]
Can show: if \( T(w_1, w_2) \) is zero plane and \( H(w) \) is zero plane

Then \( \rightarrow \) \( H(w_1, w_2) \) will be zero plane.

Suppose \( T(w_1, w_2) \) is freq. response

\[
t(n_1, n_2) < (2M+1) \times (2M+1)
\]

\[
H(w) < 2N+1 \text{ point.}
\]

\[
T(w_1, w_2) = \sum_{n_1=-M}^{+M} \sum_{n_2=-M}^{+M} t(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}
\]
\[ H(\omega_1, \omega_2) = \sum_{n=0}^{N} b(n) \left( \sum_{n_1=-M}^{+M} \sum_{n_2=-M}^{+M} t(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \right) \]

\[ H(\omega_1, \omega_2) = \sum_{n_1=-NM}^{+NM} \sum_{n_2=-NM}^{+NM} h(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \]

\[ (2NM+1) \times (2NM+1) \]

\[ M=1 \rightarrow t(n_1, n_2) : 3 \times 3 \]

\[ N=10 \rightarrow h(n) : 21 \times 1 \]
Steps for 2D Filter

Using Transformation

1. Start with specs in 2D
2. Either choose or design \( t(u_1, u_2) \)
3. Derive specs for \( H(w) \) from specs in 2D for \( H(u_1, u_2) \) and given \( t(u_1, u_2) \)

\[ w_s, w_p, s_s, s_p \text{ for } H(w) \]

4. Design \( H(w) \) \( \rightarrow \) \( l(n) \) (optimal filter)

5. Combine \( l(n) \) and \( t(u_1, u_2) \) to get \( h(u_1, u_2) \)

\[ h(u_1, u_2) + \begin{bmatrix} H(w) \\ L_n \\ T(w_1, w_0) \end{bmatrix} = \begin{bmatrix} H(w) \\ L_n \\ T(w_1, w_0) \end{bmatrix} \]

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Figure 4.5  Fourier transform of (a) separable window and (b) circularly symmetric window obtained from the analog rectangular window with $\tau = 8$. 
Figure 4.6  Support region of $w(n_1, n_2)$ for $\tau = 8$. (a) Separable window; (b) circularly symmetric window.
Figure 4.7 Fourier transform of circularly symmetric windows with $\tau = 8$. (a) Hamming window; (b) Kaiser window with $\alpha = 1$; (c) Kaiser window with $\alpha = 3$. 
Figure 4.8  Frequency responses of lowpass filters designed by the window method. The desired impulse response was obtained by using (4.10a) with $R = 0.4\pi$. The 1-D Kaiser window was used. The support regions of the windows are those shown in Figure 4.6. Both perspective and contour plots are shown. (a) Separable window design; (b) rotated circularly symmetric window design.
Figure 4.9  Frequency responses of bandpass filters designed by the window method. The desired impulse response was obtained by using (4.10c) with $R_1 = 0.3\pi$ and $R_2 = 0.7\pi$. The 1-D Kaiser window was used. The support regions of the windows are those shown in Figure 4.6. Both perspective and contour plots are shown. (a) Separable window design; (b) rotated circularly symmetric window design.
Figure 4.9 (continued)
Figure 4.10  Example of a $15 \times 15$-point lowpass filter designed by the frequency sampling method. (a) Passband (filled-in dots), transition band (marked by “x”), and stopband (open dots) samples used in the design; (b) perspective plot of the filter frequency response; (c) contour plot of the filter frequency response.
Figure 4.10 (continued)
Figure 4.11  Example of a $15 \times 15$-point bandpass filter designed by the frequency sampling method. Transition regions used in the design are from $0.3\pi$ to $0.4\pi$ and from $0.7\pi$ to $0.8\pi$. (a) Perspective plot of the filter frequency response; (b) contour plot of the filter frequency response.