Introduction to Wavelet

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Outline of Talk

- Overview
- Historical Development
- Time vs Frequency Domain Analysis
- Fourier Analysis
- Fourier vs Wavelet Transforms
- Wavelet Analysis
- Typical Applications
- References
OVERVIEW

- **Wavelet**
  - A small wave

- **Wavelet Transforms**
  - Convert a signal into a series of wavelets
  - Provide a way for analyzing waveforms, bounded in both frequency and duration
  - Allow signals to be stored more efficiently than by Fourier transform
  - Be able to better approximate real-world signals
  - Well-suited for approximating data with sharp discontinuities

- "The Forest & the Trees"
  - Notice gross features with a large "window"
  - Notice small features with a small
Historical Development

- Pre-1930
  - Joseph Fourier (1807) with his theories of frequency analysis
- The 1930s
  - Using scale-varying basis functions; computing the energy of a function
- 1960-1980
  - Guido Weiss and Ronald R. Coifman; Grossman and Morlet
- Post-1980
  - Stephane Mallat; Y. Meyer; Ingrid Daubechies; wavelet applications today
Mathematical Transformation

- Why
  - To obtain a further information from the signal that is not readily available in the raw signal.

- Raw Signal
  - Normally the time-domain signal

- Processed Signal
  - A signal that has been "transformed" by any of the available mathematical transformations

- Fourier Transformation
  - The most popular transformation
FREQUENCY ANALYSIS

- Frequency Spectrum
  - Be basically the frequency components (spectral components) of that signal
  - Show what frequencies exists in the signal

- Fourier Transform (FT)
  - One way to find the frequency content
  - Tells how much of each frequency exists in a signal

\[
X(k+1) = \sum_{n=0}^{N-1} x(n+1) \cdot W_N^{kn}
\]

\[
x(n+1) = \frac{1}{N} \sum_{k=0}^{N-1} X(k+1) \cdot W_N^{-kn}
\]

\[
w_N = e^{-j\frac{2\pi}{N}}
\]

\[
X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-2\pi jft} \, dt
\]

\[
x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{2\pi jft} \, df
\]
Stationary Signal
- Signals with frequency content unchanged in time
- All frequency components exist at all times

Non-stationary Signal
- Frequency changes in time
- One example: the “Chirp Signal”
2 Hz + 10 Hz + 20Hz

Stationary

0.0-0.4: 2 Hz + 0.4-0.7: 10 Hz + 0.7-1.0: 20Hz

Non-Stationary
**CHIRP SIGNALS**

Frequency: 2 Hz to 20 Hz

**Different in Time Domain**

Frequency: 20 Hz to 2 Hz

**Same in Frequency Domain**

At what time do the frequency components occur? FT cannot tell!
NOTHING MORE, NOTHING LESS

- FT Only Gives what Frequency Components Exist in the Signal
- The Time and Frequency Information can not be Seen at the Same Time
- Time-frequency Representation of the Signal is Needed

Most of Transportation Signals are Non-stationary.
(We need to know whether and also when an incident was happened.)

ONE EARLIER SOLUTION: SHORT-TIME FOURIER TRANSFORM (STFT)
**SФОРТ TIME FOURIER TRANSFORM (STFT)**

- Dennis Gabor (1946) Used STFT
  - To analyze only a small section of the signal at a time
  -- a technique called *Windowing the Signal*.
- The Segment of Signal is Assumed *Stationary*
- A 3D transform

\[
STFT_x^{(\omega)}(t', f) = \int [x(t) \cdot \omega^* (t - t')] \cdot e^{-j2\pi f t'} dt
\]

\(\omega(t)\): the window function

A function of time and frequency
DRAWBACKS OF STFT

Unchanged Window
- Dilemma of Resolution
  - Narrow window -> poor frequency resolution
  - Wide window -> poor time resolution
- Heisenberg Uncertainty Principle
  - Cannot know what frequency exists at what time intervals

Via Narrow Window

Via Wide Window
MULTIRESOLUTION ANALYSIS (MRA)

- Wavelet Transform
  - An alternative approach to the short time Fourier transform to overcome the resolution problem
  - Similar to STFT: signal is multiplied with a function

- Multiresolution Analysis
  - Analyze the signal at different frequencies with different resolutions
  - Good time resolution and poor frequency resolution at high frequencies
  - Good frequency resolution and poor time resolution at low frequencies
  - More suitable for short duration of higher frequency; and longer duration of lower frequency components
PRINCIPLES OF WAELET TRANSFORM

- Split Up the Signal into a Bunch of Signals
- Representing the Same Signal, but all Corresponding to Different Frequency Bands
- Only Providing What Frequency Bands Exists at What Time Intervals
DEFINITION OF CONTINUOUS WAVELET TRANSFORM

CWT \( x (\tau, s) = \Psi_x^\psi (\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \ast \frac{\psi(t - \tau)}{s} \, dt \)

- Wavelet
  - Small wave
  - Means the window function is of finite length

- Mother Wavelet
  - A prototype for generating the other window functions
  - All the used windows are its dilated or compressed and shifted versions
SCALE

- Scale
  - $S>1$: dilate the signal
  - $S<1$: compress the signal

- Low Frequency -> High Scale -> Non-detailed Global View of Signal -> Span Entire Signal

- High Frequency -> Low Scale -> Detailed View Last in Short Time

- Only Limited Interval of Scales is Necessary
COMPUTATION OF CWT

\[
CWT_x^\psi (\tau, s) = \Psi_x^\psi (\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \cdot \psi^* \left( \frac{t - \tau}{s} \right) dt
\]

**Step 1:** The wavelet is placed at the beginning of the signal, and set \( s=1 \) (the most compressed wavelet);
**Step 2:** The wavelet function at scale “1” is multiplied by the signal, and integrated over all times; then multiplied by \( \frac{1}{\sqrt{|s|}} \);
**Step 3:** Shift the wavelet to \( t=\tau \), and get the transform value at \( t=\tau \) and \( s=1 \);
**Step 4:** Repeat the procedure until the wavelet reaches the end of the signal;
**Step 5:** Scale \( s \) is increased by a sufficiently small value, the above procedure is repeated for all \( s \);
**Step 6:** Each computation for a given \( s \) fills the single row of the time-scale plane;
**Step 7:** CWT is obtained if all \( s \) are calculated.
RESOLUTION OF TIME & FREQUENCY

- Each box represents an equal portion
- Resolution in STFT is selected once for entire analysis
COMPARISON OF TRANSFORMATIONS

From http://www.cerm.unifi.it/EUcourse2001/Gunther_lecturenotes.pdf, p.10
DISCRETIZATION OF CWT

- It is Necessary to Sample the Time-Frequency (scale) Plane.
- At High Scale $s$ (Lower Frequency $f$), the Sampling Rate $N$ can be Decreased.
- The Scale Parameter $s$ is Normally Discretized on a Logarithmic Grid.
- The most Common Value is 2.
- The Discretized CWT is not a True Discrete Transform

- Discrete Wavelet Transform (DWT)
  - Provides sufficient information both for analysis and synthesis
  - Reduce the computation time sufficiently
  - Easier to implement
  - Analyze the signal at different frequency bands with different resolutions
  - Decompose the signal into a coarse approximation and detail information
Multi Resolution Analysis

- Analyzing a signal both in time domain and frequency domain is needed many a times
  - But resolutions in both domains is limited by Heisenberg uncertainty principle
- Analysis (MRA) overcomes this, how?
  - Gives good time resolution and poor frequency resolution at high frequencies and good frequency resolution and poor time resolution at low frequencies
  - This helps as most natural signals have low frequency content spread over long duration and high frequency content for short durations
**SUBBABD CODING ALGORITHM**

- Halves the Time Resolution
  - Only half number of samples resulted
- Doubles the Frequency Resolution
  - The spanned frequency band halved

0-1000 Hz

X[n]
512

Filter 1

256

D₁: 500-1000 Hz

Filter 2

256

128

D₂: 250-500 Hz

Filter 3

128

64

D₃: 125-250 Hz

A₃: 0-125 Hz
RECONSTRUCTION

What
- How those components can be assembled back into the original signal without loss of information?
- A Process After decomposition or analysis.
- Also called synthesis

How
- Reconstruct the signal from the wavelet coefficients
- Where wavelet analysis involves filtering and down sampling, the wavelet reconstruction process consists of up sampling and filtering
WAVELET APPLICATIONS

- Typical Application Fields
  - Astronomy, acoustics, nuclear engineering, sub-band coding, signal and image processing, neurophysiology, music, magnetic resonance imaging, speech discrimination, optics, fractals, turbulence, earthquake-prediction, radar, human vision, and pure mathematics applications

- Sample Applications
  - Identifying pure frequencies
  - De-noising signals
  - Detecting discontinuities and breakdown points
  - Detecting self-similarity
  - Compressing images
REFERENCES

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- Robi Polikar, Multiresolution Wavelet Analysis of Event Related Potentials for the Detection of Alzheimer’s Disease, Iowa State University, 06/06/1995
Thank You