Huffman Coding: A practical method to achieve near-optimum performance

Example:

<table>
<thead>
<tr>
<th>Message</th>
<th>Codeword</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁</td>
<td>0</td>
<td>( \frac{5}{8} )</td>
</tr>
<tr>
<td>a₂</td>
<td>100</td>
<td>( \frac{3}{32} )</td>
</tr>
<tr>
<td>a₃</td>
<td>110</td>
<td>( \frac{3}{32} )</td>
</tr>
<tr>
<td>a₄</td>
<td>1110</td>
<td>( \frac{1}{32} )</td>
</tr>
<tr>
<td>a₅</td>
<td>101</td>
<td>( \frac{1}{8} )</td>
</tr>
<tr>
<td>a₆</td>
<td>1111</td>
<td>( \frac{1}{16} )</td>
</tr>
</tbody>
</table>

Figure 10.16 Illustration of codeword generation in Huffman coding. Message possibilities with higher probabilities are assigned with shorter codewords.

- Uniform-length codeword: 3 bits/message
- Huffman coding: \( \frac{5}{8} \cdot 1 + \frac{3}{32} \cdot 3 + \frac{3}{32} \cdot 3 + \frac{1}{16} \cdot 4 + \frac{1}{8} \cdot 3 + \frac{1}{16} \cdot 4 = \frac{77}{16} \) bits/message
- Entropy: \(- \left( \frac{5}{8} \log_2 \frac{5}{8} + \frac{3}{32} \log_2 \frac{3}{32} + \frac{3}{32} \log_2 \frac{3}{32} + \frac{1}{16} \log_2 \frac{1}{16} + \frac{1}{8} \log_2 \frac{1}{8} + \frac{1}{16} \log_2 \frac{1}{16} \right) = 1.752 \) bits/message
Methods of Bit Assignment (cont.)

Vector Case

<table>
<thead>
<tr>
<th>$f_1, f_2$</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0, r_0'$</td>
<td>$p_0$</td>
</tr>
<tr>
<td>$r_1, r_1'$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>$r_2, r_2'$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Entropy = $- \sum_{i=0}^{N-1} p_i \cdot \log_2 p_i$

Comments:

1. Finding $p_i$ for a large number of elements in the vector is difficult
2. Law of diminishing returns comes into play
3. Huffman Coding can be used
Arithmetic Coding

- Why not use Huffman?

- Can show rate of Huffman code is within $P_{max} + 0.086$ of Entropy.

  $$P_{max} = \text{Prob. of most frequent symbol}$$

- Example:

  iid source = \{a_1, a_2, a_3\}

  $$P(a_1) = 0.95 \quad P(a_2) = 0.02 \quad P(a_3) = 0.03$$

  Entropy = 0.335 bits/symbol

  Huffman : 1.05 bits/symbol

  $$\implies \text{Huffman 213\% of Entropy!!!}$$
Mechanics of Arithmetic Coding

• Two steps:
  1) Generate a tag
  2) Assign a binary code to the tag

• Tag Generation:
  A sequence of symbols from an alphabet source with a given Pdf results in a unique sub-interval in [0,1].
Example of Tag Generation:

- $A = \{a_1, a_2, a_3\}$  \hspace{1cm} $P(a_1) = 0.7$
  \hspace{1cm} $P(a_2) = 0.1$  \hspace{1cm} $P(a_3) = 0.2$

- Suppose we encode $a_1, a_2, a_3$

- Interval $[0.546, 0.56]$ uniquely specifies sequence $(a_1, a_2, a_3)$
Deciphering the Tag:

- Suppose we know 3 symbols resulted in subinterval \([0.546, 0.56]\)

- Decode the symbols?

- Is the first symbol \(a_1\) or \(a_2\) or \(a_3\)?

\[
egin{align*}
  a_1 & \rightarrow [0, 0.7] \\
  a_2 & \rightarrow [0.7, 0.8] \\
  a_3 & \rightarrow [0.8, 1] \\
\end{align*}
\]

\(\Rightarrow \ a_1 \Rightarrow \)

- Is the second symbol \(a_1, a_2, \) or \(a_3\)?

\[
egin{align*}
  a_1 & \rightarrow [0, 0.49] \\
  a_2 & \rightarrow [0.49, 0.56] \\
  a_3 & \rightarrow [0.56, 0.7] \\
\end{align*}
\]

\(\Rightarrow \ a_2 \Rightarrow \)
Generating a binary code:

- Uniqueness:
- Consider sequence of symbols
  \[ \vec{X} = a_1, a_3, a_2, \ldots, a_1, \ldots a_3, a_2. \]
- Compute \( P(\vec{X}) \).
  If iid source ==> easy to compute
- Write down binary representation of the point in the middle of subinterval for \( \vec{X} \)
- Truncate it to
  \[
  \left\lfloor \log \frac{1}{P(\vec{X})} \right\rfloor + 1 \text{ bits}
  \]
- Can show it is unique.
Example Generating a binary code:

- Consider subinterval \([0.546, 0.56]\)
- Midpoint : 0.553

\[ P(\vec{x}) = P(a_1)P(a_2)P(a_3) = 0.014 \]

- \# of bits = \( \left\lfloor \log \frac{1}{P(\vec{x})} \right\rfloor + 1 = 8 \) bits

- Binary Representation of 0.553

\[
0.553 = \frac{1}{2^1} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^9} + \ldots
\]

\[
= 0.1 \ 00011011\ldots
\]

- Truncate to 8 bits ===>
  Binary Representation:

\[
10001101
\]
Efficiency of Arithmetic Code

- We can show

\[ H(X) \leq l_A \leq H(X) + \frac{2}{m} \]

- \( m \) # of symbols in sequence
- \( l_A \) = average length per symbol
- \( H(X) \) = entropy of source
- Assumes iid source.

- Observe: By increasing length of sequence, can get arbitrarily close to entropy.
Dictionary Techniques

- Huffman and arithmetic coding assumed i.i.d. sources

- Most sources are correlated

- Main idea: 1. Build a list of commonly occurring patterns
  2. Transmit index in the list.

- Exploits fact that certain patterns recur frequently.

- Consider 2 cases
  1. Static Dictionary
  2. Dynamic Dictionary
Static Dictionary:

- Suppose Five letter alphabet source:

\[ A = \{ a, b, c, d, r \} \]

- Using statistics of source, build dictionary

<table>
<thead>
<tr>
<th>Entry</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>ad</td>
<td>111</td>
</tr>
<tr>
<td>ac</td>
<td>110</td>
</tr>
<tr>
<td>ab</td>
<td>101</td>
</tr>
<tr>
<td>r</td>
<td>100</td>
</tr>
<tr>
<td>d</td>
<td>011</td>
</tr>
<tr>
<td>c</td>
<td>010</td>
</tr>
<tr>
<td>b</td>
<td>001</td>
</tr>
<tr>
<td>a</td>
<td>000</td>
</tr>
</tbody>
</table>

- Encode a b r a c a d a b r a

1. Read ab ----------> 101
2. Read ra ----------> not in dictionary
3. Read r ----------> 100
4. Read ac ----------> \emptyset 110
5. ........

101 1\text**0** 110 111 101 101 000
ab r ac ad ab r a

- Opposite of Huffman coding:
Adaptive Dictionary:

- Ziv Lempel 1977 + 1978
- LZ1 ----> 1977  LZ2 ----> 1978
- LZ1 discussed Here.

- Basic idea:
  
  Dictionary portion of previously encoded sequence

- Sliding window:
  1) search buffer
  2) lookahead buffer

match pointer

```
c x x a b p a x a d a b a p p a p p a
```

search buffer  Look ahead buffer
Example

...c abracadabrararrad ...

- Window = 13
- look ahead buffer = 6
- Search buffer = 7

...c abracadabrararrad ...

1. No match to d ---> <φ, 0, C(d)>

...cabracadabrararrad ...

2. Match for a: 
\[
\begin{align*}
\emptyset &= 2 \quad \Rightarrow \quad l = 1 \\
\emptyset &= 4 \quad \Rightarrow \quad l = 1 \\
\emptyset &= 7 \quad \Rightarrow \quad l = \boxed{4}
\end{align*}
\]

< 7, 4, C(r) >

...adabrararrad ...
3. Mach for $r$:

\[
\begin{align*}
0 &= 1 \quad \rightarrow \quad l = 1 \\
0 &= 3 \quad \rightarrow \quad l = 5
\end{align*}
\]

\(< 3, 5, C(d) >\)

exceeds the boundary between search and look ahead buffer
**Encoding steps**

1. Move search pointer back until match in search buffer

2. Offset \( \triangleq \) Distance of pointer from lookahead buffer

3. Do consecutive symbols of pointer match also?

4. Search the search buffer for the longest match

5. length of match \( \triangleq \) # of consecutive symbol match.

6. Send \( < o, l, c > \)
   
   \( o \) = offset
   
   \( l \) = match length
   
   \( c \) = codeword of symbol in LA buffer, following match

**Example:** \( < 7, 2, \text{codeword for a} > \)
Adaptive Dictionary

- Why send $C$?
  - just in case no match

- Total # of bits:

$$\lceil \log_2 S \rceil + \lceil \log_2 W \rceil + \lceil \log_2 A \rceil$$

$S$ = size of search buffer

$W$ = Size of window (search + LA)

$A$ = size of source alphabet
What to Code (Classification of Image Coding Systems)

1. Waveform Coder (code the intensity)
   - PCM (Pulse Code Modulation) and its improvements
   - DM (Delta Modulation)
   - DPCM (Differential Pulse Code Modulation)
   - Two-channel Coder

2. Transform Coder (code transform coefficients of an image)
   - Karhunen-Loeve Transform
   - Discrete Fourier Transform
   - Discrete Cosine Transform

3. Image Model Coder
   - Auto-regressive Model for texture
   - Modelling of a restricted class of images

Note: Each of the above can be made to be adaptive
Waveform Coder

PCM Coding

\[ f(n, n_2) \rightarrow \text{Uniform Quantizer} \rightarrow \hat{f}(n, n_2) \]

\[ \hat{f}(n, n_2) \rightarrow \text{Non-Linearity} \rightarrow \text{Uniform Quantizer} \rightarrow \text{Non-Linearity}^{-1} \rightarrow \hat{f}(n, n_2) \]

- very simple
- typically requires over 5-6 bits/pixel for good quality
- false contours for low-bit rate case
Improvements of PCM (cont.)

2. Roberts' Pseudo-Noise Technique with Noise Reduction:

Transmitter

\[ f(n_1, n_2) \]

\[ \rightarrow \]

\[ + \]

\[ \rightarrow \]

Uniform quantizer

\[ \rightarrow \]

\[ + \]

\[ \rightarrow \]

Noise reduction

\[ \rightarrow \]

\[ f(n_1, n_2) \]

\[ \rightarrow \]

\[ w(n_1, n_2) \]

\[ \rightarrow \]

\[ w(n_1, n_2) \]

3. Roberts' Pseudo-Noise Technique and Highpass/Lowpass Filtering:

\[ f(n, n_2) \rightarrow \]

\[ \rightarrow \]

\[ \rightarrow \]

Uniform quantizer

\[ \rightarrow \]

\[ \rightarrow \]

Low-pass

\[ \rightarrow \]

\[ f(n, n_2) \]

If there were not Highpass/Lowpass Filtering:

\[ \overline{F} \]

\[ \rightarrow \]

\[ \overline{F_1} \]

\[ \rightarrow \]

\[ \overline{F_a} \]

\[ \rightarrow \]

\[ \overline{F_1'} \]
Improvements of PCM

1. Roberts' Pseudo-Noise Technique:

- false contours disappear — replaced by additive random noise
Figure 10.21 Example of quantization noise reduction in PCM speech coding. (a) Segment of noise-free voiced speech; (b) PCM-coded speech at 2 bits/sample; (c) PCM-coded speech at 2 bits/sample by Roberts's pseudonoise technique; (d) PCM-coded speech at 2 bits/sample with quantization noise reduction.
Figure 10.22 Example of quantization noise reduction in PCM image coding. (a) Original image of 512 x 512 pixels; (b) PCM-coded image at 2 bits per pixel; (c) PCM-coded image at 2 bits per pixel by Roberts's pseudonoise technique; (d) PCM-coded image at 2 bits per pixel with quantization noise reduction.
Delta Modulation (DM)

\[ f(n_1, n_2) : \text{signal}, \quad \hat{f}(n_1, n_2) : \text{coded signal} \]

Transmitter

\[ f(n) + e(n) \xrightarrow{1 \text{ bit quantization}} \hat{e}(n) = \frac{\Delta}{2} \text{ or } -\frac{\Delta}{2} \]

\[ \hat{f}(n - 1) \rightarrow \hat{f}(n) \]

Receiver

\[ \hat{e}(n) + \hat{f}(n - 1) \rightarrow \hat{f}(n) \]

Figure 10.26 Granular noise and slope-overload distortion in delta modulation.

- needs over 2-3 bits/pixel to get good quality
quirements, and the step size \( \Delta \) is chosen through some compromise between the two requirements.

Figure 10.27 illustrates the performance of a DM system. Figures 10.27(a) and (b) show the results of DM with step sizes of \( \Delta = 8\% \) and \( 15\% \), respectively, of the overall dynamic range of \( f(n_1, n_2) \). The original image used is the 512 \( \times \) 512-pixel image in Figure 10.22(a). When \( \Delta \) is small [Figure 10.27(a)], the granular noise is reduced, but the slope overload distortion problem is severe and the resulting image appears blurred. As we increase \( \Delta \) [Figure 10.27(b)], the slope overload distortion is reduced, but the graininess in the regions where the signal varies slowly is more pronounced.

![Granular noise and slope overload distortion in delta modulation](image)

**Figure 10.26** Granular noise and slope overload distortion in delta modulation.

**Figure 10.27** Example of delta-modulation DM-coded image. The original image used is the image in Figure 10.22(a). (a) DM-coded image with \( \Delta = 8\% \) of the overall dynamic range. NMSE = 14.87, SNR = 8.3 dB. (b) DM-coded image with \( \Delta = 15\% \). NMSE = 9.77, SNR = 10.1 dB.
To obtain good quality image reconstruction using DM without significant graininess or slope overload distortion, 3–4 bits/pixel is typically required. A bit rate higher than 1 bit/pixel can be obtained in DM by oversampling the original analog signal relative to the sampling rate used in obtaining $f(n, n_2)$. Oversampling reduces the slope of the digital signal $f(n)$ so a smaller $\Delta$ can be used without increasing the slope overload distortion. An example of an image coded by DM at 2 bits/pixel is shown in Figure 10.28. To obtain the image in Figure 10.28, the size of the original digital image in Figure 10.22(a) was increased by a factor of two by interpolating the original digital image by a factor of two along the horizontal direction. The interpolated digital image was coded by DM with $\Delta = 12\%$ of the dynamic range of the image and the reconstructed image was undersampled by a factor of two along the horizontal direction. The size of the resulting image is the same as the image in Figure 10.27, but the bit rate in this case is 2 bits/pixel.

### 10.3.3 Differential Pulse Code Modulation

Differential pulse code modulation (DPCM) can be viewed as a generalization of DM. In DM, the difference signal $e(n) = f(n) - f(n - 1)$ is quantized. The most recently coded $f(n - 1)$ can be viewed as a prediction of $f(n)$ and $e(n)$ can be viewed as the error between $f(n)$ and a prediction of $f(n)$. In DPCM, a prediction of the current pixel intensity is obtained from more than one previously coded pixel intensity. In DM, only one bit is used to code $e(n)$. In DPCM, more than one bit can be used in coding the error.

A DPCM system is shown in Figure 10.29. To code the current pixel intensity $f(n_1, n_2)$, $f(n_1, n_2)$ is predicted from previously reconstructed pixel intensities. The predicted value is denoted by $f'(n_1, n_2)$. In the figure, we have assumed that $f(n_1 - 1, n_2), f(n_1, n_2 - 1), f(n_1 - 1, n_2 - 1), \ldots$ were reconstructed prior to coding $f(n_1, n_2)$. We are attempting to reduce the variance of...
Differential Pulse Code Modulation (DPCM)

\[ f(n_1, n_2) : \text{original image} \]
\[ \hat{f}(n_1, n_2) : \text{reconstructed image} \]

**Transmitter**

- the Auto-regressive Model parameters are obtained from the image by solving a linear set of equations or by a Markov process assumption

**Receiver**

- requires 2-3 bits/pixel for good quality image
where $R_a$ is the region of $(k_1, k_2)$ over which $a(k_1, k_2)$ is nonzero. Typically, $\hat{f}(n_1, n_2)$ is obtained by linearly combining $\hat{f}(n_1 - 1, n_2), \hat{f}(n_1, n_2 - 1)$, and $\hat{f}(n_1 - 1, n_2 - 1)$. Since the prediction of $f(n_1, n_2)$ is made in order to reduce the variance of $e(n_1, n_2)$, it is reasonable to estimate $a(k_1, k_2)$ by minimizing

$$E[e^2(n_1, n_2)] = E[(\hat{f}(n_1, n_2) - \sum_{(k_1, k_2) \in R_a} a(k_1, k_2)f(n_1 - k_1, n_2 - k_2))^2].$$

(10.40)

Since $\hat{f}(n_1, n_2)$ is a function of $a(k_1, k_2)$ and depends on the specific quantizer used, solving (10.40) is a nonlinear problem. Since $\hat{f}(n_1, n_2)$ is the quantized version of $f(n_1, n_2)$, and is therefore a reasonable representation of $f(n_1, n_2)$, the prediction coefficients $a(k_1, k_2)$ are estimated by minimizing

$$E\left[ (f(n_1, n_2) - \sum_{(k_1, k_2) \in R_a} a(k_1, k_2)f(n_1 - k_1, n_2 - k_2))^2 \right].$$

(10.41)

Since the function in (10.41) minimized is a quadratic form of $a(k_1, k_2)$, solving (10.41) involves solving a linear set of equations in the form of

$$R_i(l_1, l_2) = \sum_{(k_1, k_2) \in R_a} a(k_1, k_2)R_i(l_1 - k_1, l_2 - k_2)$$

(10.42)

where $f(n_1, n_2)$ is assumed to be a stationary random process with the correlation function $R_i(n_1, n_2)$. The linear equations in (10.42) are the same as those used in the estimation of the autoregressive model parameters discussed in Chapters 5 and 6.

Figure 10.30 illustrates the performance of a DPCM system. Figure 10.30 shows the result of a DPCM system at 3 bits/pixel. The original image used is the image in Figure 10.22(a). The PCM system used in Figure 10.30 is a nonuniform quantizer. The prediction coefficients $a(k_1, k_2)$ used to generate the example are

Figure 10.30 Example of differential pulse code modulation (DPCM)-coded image at 3 bits/pixel. Original image used is the image in Figure 10.22(a). NMSE $= 2.25$, SNR $= 16.8$ dB.