Reconstruction of 2-D Signals
from Partial Fourier Information

- Motivation: electron microscopy, optical astronomy, cryo-electron microscopy...

- 3 Problems:
  1. FTM = magnitude of Fourier Transform
  2. Reconstruct Phase Fourier Transform
  3. Reconstruct Level Odometry
1-D Case:
- $X(n)$: discrete time 1-D
- $n = 0, \ldots, N$

D.T.F.T. $\{x(n)\} = X(w) = \sum_{n=0}^{N} x(n)e^{-jwn}$

If we know $|X(w)|$, can we get $x(n)$?

If I know $|X(w)|$ at $N+1$ points, cannot reconstruct signal.
Sample a 2-D poly edge at arbitrary points to result in a unique recon of the polynomial.

\[ \sum_{n=0}^{N} x(n) y^n = X(\omega) \]
Q: Can we uniquely reconstruct if we oversampled F.D.?

A: For 1D signal \( \rightarrow \) No \( 2^p \) ambiguity

MD signal \( \rightarrow \) yes
auto-correlation fn of $x(n)$

$$r(n) = \sum_{l} x(l) x^*(l+n)$$

$$R(z) = D.T.F.T \{ r(n) \} = \sum_{n=-\infty}^{\infty} r(n) z^{-n} = |X(z)|^2$$

$$\Rightarrow F^{-1} \{ |X(z)|^2 \} = r(n)$$

Q: Can we obtain $x(n)$ from

$$r(n) = \sum_{n=-\infty}^{\infty} r(n) z^{-n}$$

Z.T. $\{ r(n) \} \Rightarrow R(z) = X(z) X^*(\frac{1}{z^*}) = \sum_{n=-N}^{N} r(n) z^{-n}$

Associated polynomial to $R(z) \triangleq P_r(z) = \sum_{n=0}^{2N} r[N-n] z^n$

Since $P_r(z)$ is symmetric, $z$ is a zero of $P_r(z)$, so is $z^*$. If $\theta_0$ is a zero of $P_r(z)$, so is $\theta_0 + 1$. If $\theta_0$ is a zero of $P_r(z)$, so is $\theta_0$. If $\theta_0$ is a zero of $P_r(z)$, so is $\theta_0$. If $\theta_0$ is a zero of $P_r(z)$, so is $\theta_0$.
Can factor \( P_r(z) \)

\[
P_r(z) = A \prod_{i=1}^{N} \left( z - z_i \right) \left( 1 - \frac{z_i}{z} \right)
\]

**Def mirror of a polynomial.**

Associated with any polynomial, there is a mirror polynomial consisting of coefficients in reverse order and conjugated.

If \( P(z) = \sum_{n=0}^{N} p_n z^n \rightarrow \hat{P}(z) = \sum_{n=0}^{N} \ast p_{N-n} z^n \)
- Assume $y(n)$ auto correlation $r(n)$

\[ r(z) = P_y(z) \frac{z^{-1}}{P_y(z)} \]

$2^N$ ways in order to generate $P_y(z)$.

\[ P_y(z) = \sum_{i \in S} \prod_{j \in S} (z - z_{ij}) \prod_{j \notin S} (z - z_{ij}^*) \]

any subset of $[1, \ldots, N]$
\[
\begin{align*}
(z_0, z_0^* - 1) & \quad P(z) \\
(z_1, z_1^* - 1) & \quad P_r(z) \\
& \quad \vdots \\
(z_N, z_N^* - 1) & \quad \Gamma(n) \\
\end{align*}
\]

\[
2^N \cdot x(n) \quad \rightarrow \quad \text{same } r(n)
\]
2-D case FM

Assume $|X(w_1, w_2)|^2$

$x(n_1, n_2) : N \times N$

$\Rightarrow$ in 2-D, unlike 1-D, most 2D polynomials are irreducible $\Rightarrow$ non-factorable.
If $x(n_1,n_2)$ has irreducible associated polynomials, then all other $y(n)$ which have same F.T.M are equivalent to $x$. Equivalent mean:

$$y(n_1,n_2) = x(n_1,n_2)$$

if

$$y(n_1,n_2) = e^{j\theta} x(n_1,n_2)$$

Observation 1: $x(n_1,n_2) \rightarrow |x(w_1,w_2)|^2$

$$e^{j\theta} x(n_1,n_2) \rightarrow |x(w_1,w_2)|^2$$

Observation 2: $x(n_1,n_2) \rightarrow |x(w_1,w_2)|^2$

Shift by $k_1,k_2$ in space domain $\rightarrow$ same F.T.M
Observation 3: Rotational Sector

\[ x(n_1, n_2) \rightarrow |X(\omega_1, \omega_2)|^2 \]

\[ y(n_1, n_2) = x(N-n_1, N-n_2) \]
also has same FTMs.

- Assume signal \( x(n_1, n_2) \) is real
  \[ e^{j\theta} \]
  \[ \theta = 0 \text{ or } \theta = \pi \]
- Assume signal is positive \( \Rightarrow \) \( \theta = 0 \)
2. Assumption: extent of \( x(\nu_1, \nu_2) \) is known.

3. 2.T of signal \( x \rightarrow \) irreducible.

Hence: nail signal to factor of 2:

either \( x(\nu_1, \nu_2) \) or \( x(N-\nu_1, N-\nu_2) \)

Same FTM.

\( x \) as real, written and positive.

Know extent.

FTM can be used to recover either \( x \)
or its reflection "uniquely"
Assume known 4N × 4N sample of \( x(w_1, w_2) \).

\[ \begin{align*}
\text{NYW} & : \, 0 \rightarrow \frac{N}{2} - 1 \\
\text{NYW} & : \, 0 \rightarrow \frac{N}{2} - 1 \\
x \text{ is positive real}
\end{align*} \]

Proposed Iterative Alg

1. Initial guess \( N \times N \)
2. DFT \( 4N \times 4N \)
3. Reality may same as given?
4. Yes → STOP
5. No → \( 4N \times 4N \)
6. Keep phase
7. Replace Mag with what is given
8. I DFT \( 4N \times 4N \)
9. Set extra coeff to zero

Show Fig. 3.2
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