Reconstruction of 2-D signals
from Partial Fourier Information

- Motivation: electron microscopy,
optical astronomy,
cryptology.

- 3 Problems:
  1. FTM = magnitude of Fourier Transform
  2. Reconstruct from Phase
     Fourier Transform
  3. Reconstruct from level
**1-D Case:**

- $X(n)$: discrete time 1-D.
- $N+1$ nonzero points: $n = 0, \ldots, N$

D.T.F.T $\left\{ x(n) \right\} = X(w) = \sum_{n=0}^{N} x(n)e^{-jwn}$

**IF WE KNOW** $|X(w)|$ **CAN WE GET** $x(n)$?

**IF I KNOW** $|X(w)|$ **AT** $N+1$ **POINTS** $\rightarrow$ **CANNOT RECON SIGNAL**
\[ x(n) \sum_{n=0}^{N} x(n) y^n = X(w) \]

Sample a
2-D poly edge
at arbitrary
point
result
in unique
recon of
polynomial
Q: Can we uniquely reconstruct if we oversampled F.D.?

A: for 1D signal → No ambiguity
for MD signal → Yes
auto-correlation fn of \( x(n) \):
\[
    r(n) = \sum_{l} x(l) x^*(l+n)
\]

\[
    R(\omega) = \text{D.T.F.T} \{ r(n) \} = \sum_{n} r(n) e^{-j\omega n} = |X(\omega)|^2
\]

\[
    \implies \mathcal{F}^{-1} \{ |X(\omega)|^2 \} = r(n)
\]

Q: Can we obtain \( x(n) \) from \( r(n) \)?

\[
    \implies \{ r(n) \} = R(z) = X(z) X^* \left( \frac{1}{z^*} \right) = \sum_{n=-N}^{+N} r(n) z^{-n}
\]

Associated polynomial to \( R(z) \) is \( P_r(z) = \sum_{n=0}^{2N} r[N-n] z^n \)

Since \( P_r(z) \) is symmetric \( \implies \) if \( z_0 \) is a zero of \( P_r(z) \), so is \( \bar{z}_0 \).

If \( z_0 \) is a zero of \( P_r(z) \), so is \( \bar{z}_0 \).
Def mirror of a polynomial.

Associated with any polynomial, there is a mirror polynomial consisting of coefficients in reverse order and conjugated.

If \( P(z) = \sum_{n=0}^{N} a_n z^n \) is the mirror of \( \tilde{P}(z) = \sum_{n=0}^{N} a_{N-n} z^{N-n} \).
- Assume \( y(n) \) \( \rightarrow \) auto correlation \( r(n) \)

\[
P_r(z) = P_y(z) P_y^*(z)
\]

\( 2^N \) way in order to generate \( P_y(z) \):

\[
P_y(z) = \sqrt{A} \prod (z - z_i) \prod (z - z_j^*)
\]

\( i \epsilon I \)

\( j \epsilon I \)

I any subset of \([1, \ldots, N]\)
2-D case F7M

Assume \( L X(w_1, w_2) \)

\[ X(n_1, n) : N \times N \]

\( \Rightarrow \) in 2-D, unlike 1-D, most 2D polynomials are irreducible \( \Rightarrow \) non-factorable.
1982, M. Heyer: If \( x(n_1, n_2) \) has irreducible associated polynomials, then all other \( y(n) \) which have same F.T.M are equivalent to \( x \). Equivalent mean:

\[
y(n_1, n_2) \sim x(n_1, n_2)
\]

\[
y(n_1, n_2) = e^{j\theta} x(k_1 + n_1, k_2 + n_2)
\]

- Observation 1: \( x(n_1, n_2) \rightarrow |X(w_1, w_2)|^2 \)

\[
j\theta
\]

\[
e^{j\theta} x(n_1, n_2) \rightarrow |X(w_1, w_2)|^2
\]

- Observation 2: \( x(n_1, n_2) \rightarrow |X(w, q)|^2 \)

Shift by \( k_1, k_2 \) in space domain \( \rightarrow \) same F.T.M
Observation 3: Rotational Sector

\[ x(n_1, n_2) \rightarrow |X(w_1, w_2)|^2 \]
\[ y(n_1, n_2) = x(N-n_1, N-n_2) \]
also has same FTM.

- Assume signal \( x(n_1, n_2) \) is real
- \( e^{j\theta} \) \( \theta = 0 \) or \( \theta = \pi \)
- Assume signal is positive \( \Rightarrow \theta = 0 \)
2. Assumption: Extent of $x(n_1,n_2)$ is known.

3. Z.T of signal $x \rightarrow$ irreducible.

Hence, nail signal to factor of 2:

Either $x(n_1,n_2)$ or $x(N-h_1,N-n_2)$

Same F.T.M.

(1) Assume $x$ real, positive and positive.

Know extent.

F.T.M can be used to recover either $x$ or its reflection “uniquely.”
Assume 'know 4N x 4N sample of $|x(w_1, w_2)|^2$ 
$N x N \rightarrow N-1$ 
$x$ is positive real

Proposed Iterative Alg

Initial guess $N x N$

$N x N \rightarrow N-1$

DFT

DFT

$4N x 4N$

$4N x 4N$

Reality may vary as given?

Yes

Keep phase

Replace Mag with what is given

No

$I$ DFT

$4N x 4N$

$4N x 4N$

Set extra code to zero

Stop

Show fig. 32

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Recon from F.T. Phase

\[ X(n_1, n_2) \xrightarrow{D.T.F.T} X(\omega_1, \omega_2) = \sum_{n_1} \sum_{n_2} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \]

\[ \Phi(\omega_1, \omega_2) \]

Patrick Van Hove \( \leq \) 1982

Need to have \( \Phi \) only at more than \( N^2 \) points.
Two Alg. → iterative.

1982 → Even quantizing phase to one bit & yet nearly signal successful.
iteration

NxN signal
Know Ros : NxN
Sample at 2Nx2N
up phase of F.7: \( \Phi_x \)

\[ y(n) = y(n+2N) \]

2Nx2N DFT
of \( y(n+2N) \)
\( Y(k_1,K_0) \)

Initial
Gren.
\( y(n) \)
NxN

Set signal to
zero outside
2Nx2N region

\( F^{-1} \) to get
another
2Nx2N space domain

Keep mag. y,
Explain phase
with give phase
2Nx2N.

\( \phi_x = \phi_y \)

\( \text{Yes} \)
\( \text{No} \)

Stop

\( x = \text{original sign} \)