EE123
Digital Signal Processing

Lecture 5C
Introduction to Wavelets
Time Dependent Fourier Transform

- To get temporal information, use part of the signal around every time point

\[ X[n, \omega] = \sum_{m=-\infty}^{\infty} x[n + m] w[m] e^{-j\omega m} \]

*Also called Short-time Fourier Transform (STFT)
Another view of STFT

- Can be expressed as a convolution

\[ X[n, \omega) = \sum_{m=-\infty}^{\infty} x[n + m]w[m]e^{-j\omega m} \]
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Basis functions (Atoms)
Heisenberg Boxes

- Time-Frequency uncertainty principle

\[ \sigma_t \cdot \sigma_\omega \geq \frac{1}{2} \]
Discrete STFT

\[ X[r, k] = \sum_{m=0}^{L-1} x[rR + m] w[m] e^{-j2\pi km/N} \]
Discrete STFT

\[ X[r, k] = \sum_{m=0}^{L-1} x[rR + m] \omega[m] e^{-j2\pi km/N} \]
Discrete STFT

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From STFT to Wavelets

• Basic Idea:
  – low-freq changes slowly - fast tracking unimportant
  – Fast tracking of high-freq is important in many apps.
  – Must adapt Heisenberg box to frequency

• Back to continuous time for a bit.....
From STFT to Wavelets

• Continuous time
From STFT to Wavelets

- **Continuous time**

\[
S_f(u, \Omega) = \int_{-\infty}^{\infty} f(t)w(t-u)e^{-j\Omega t} \, dt
\]

\[
W_f(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^\star\left(\frac{t-u}{s}\right) \, dt
\]

*Morlet - Grossmann*
From STFT to Wavelets

\[ W f(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*(\frac{t - u}{s}) dt \]

• The function \( \Psi \) is called a mother wavelet

\[ \int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1 \quad \Rightarrow \text{unit norm} \]

\[ \int_{-\infty}^{\infty} \Psi(t) dt = 0 \quad \Rightarrow \text{Band-Pass} \]
STFT and Wavelets “Atoms”

**STFT Atoms**  
(with hamming window)  
\[ w(t - u)e^{j\Omega t} \]

\[ w(t - u)e^{j\Omega t} \]

**Wavelet Atoms**  
\[ \frac{1}{\sqrt{s}} \psi\left(\frac{t - u}{s}\right) \]

\[ \frac{1}{\sqrt{s}} \psi\left(\frac{t - u}{s}\right) \]

\[ \Omega_{hi} \]

\[ \Omega_{hi} \]

\[ \Omega_{lo} \]

\[ \Omega_{lo} \]

\[ s = 1 \]

\[ s = 1 \]

\[ s = 3 \]

\[ s = 3 \]
Examples of Wavelets

- **Mexican Hat**

  \[ \Psi(t) = (1 - t^2)e^{-t^2/2} \]

- **Haar**

  \[ \Psi(t) = \begin{cases} 
    -1 & 0 \leq t < \frac{1}{2} \\
    1 & \frac{1}{2} \leq t < 1 \\
    0 & \text{otherwise} 
  \end{cases} \]
Wavelets Transform

• Can be written as linear filtering

\[ Wf(u, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} f(t) \Psi^* \left( \frac{t - u}{s} \right) dt \]

\[ = \left\{ f(t) * \overline{\Psi_s}(t) \right\} (u) \]

\[ \overline{\Psi_s} = \frac{1}{\sqrt{s}} \Psi \left( \frac{t}{s} \right) \]

• Wavelet coefficients are a result of bandpass filtering
Example 2: “Bumpy” Signal

Sombrero Wavelet

log(s)

u
Example 2: “Bumpy” Signal
Example 2: “Bumpy” Signal

$\log(s)$
Example 2: “Bumpy” Signal

Sombrero Wavelet

$\log(s)$

$u$