MULTI-Resolution Expansion

- Scaling fn \( \Phi \): create a series of approximations of a fn each differing by a factor of 2 in resolution

- Function \( \Phi \): (wavelet) encoder diff between adjacent approximations.

Series Expansion

Expand fn \( f(x) \) as:

\[
  f(x) = \sum_{k} d_k \Phi_k(x)
\]

\( \Phi_k(x) \equiv \text{real valued expansion functions} \)

\( d_k \equiv \text{coefficients} \)
If expansion unique i.e. only one set of \( a_k \) for \( f(x) \)

\[ \Rightarrow \phi_k = \text{basis function}. \quad \{ \phi_k \} = \text{basis for class of fun.} \]

Function space: \( V = \text{Span} \{ \phi_k(x) \} \) closed span of expansion set

\( f(x) \in V \implies f(x) \) is in closed span of \( \{ \phi_k(x) \} \)

and can be written as \( f(x) = \sum_k a_k \phi_k(x) \)

Dual function \( \{ \tilde{\phi}_k(x) \} \) To \( \{ \phi_k(x) \} \)

\[ a_k = \langle \tilde{\phi}_k(x), f(x) \rangle \]

Consider 3 cases:
(1) Expansion functions form an orthonormal basis for $V$:

\[
\langle \phi_j(x), \phi_k(x) \rangle = \delta_{jk} = \begin{cases} 
0 & j \neq k \\
1 & j = k
\end{cases}
\]

\[\Rightarrow \phi_k(x) = \hat{\phi}_k(x) \text{ basis and dual same.}\]

\[\Rightarrow \alpha_k = \langle \phi_k(x), f(x) \rangle\]

(2) Expansion function orthogonal but not orthonormal

\[\langle \phi_j(x), \phi_k(x) \rangle = 0 \quad j \neq k\]

\[\Rightarrow \text{basis fn and dual are bi-orthogonal}\]

\[\alpha_k = \langle \hat{\phi}_k(x), f(x) \rangle\]

\[\langle \phi_j(x), \hat{\phi}_k(x) \rangle = \delta_{jk} = \begin{cases} 
0 & j \neq k \\
1 & j = k
\end{cases}\]
3) More than one set of \( \phi_k \) in

\[
f(x) = \sum_k \alpha_k \phi_k(x)
\]

\[\Rightarrow \text{Exp fr & duals are "overcomplete" or "redundant"}
\]

Form a frame

\[
A \|f(x)\|^2 \leq \sum_k |\langle \phi_k(x), f(x) \rangle|^2 \leq B \|f(x)\|^2
\]

for \( A > 0, \ B < \infty \) \& \( f(x) \in V \)

- If \( A = B \rightarrow \text{Tight frame} \)

- Daubechies 1992 \( f(x) = \frac{1}{A} \sum_k \langle \phi_k(x), f(x) \rangle \phi_k(x) \)
Scaling functions

- Start with real, square integrable \( f \), \( \varphi(x) \)
- Build a set \( \varphi_{j,k}(x) = \frac{1}{2} \varphi(2x-k) \)
- \( j,k \in \mathbb{Z} \), \( \varphi(x) \in L^2(\mathbb{R}) \)

Denote subspace \( V_j \) : 

\[
V_j = \text{span} \{ \varphi_{j,k}(x) \}
\]

Then if \( f(x) \in V_j \) \( \Rightarrow f(x) = \sum_k d_k \varphi_{j,k}(x) \)

Example Haar basis.

\[
\varphi(x) = \begin{cases} 
1 & 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}
\]

Show Fig 7.11 6 + W 3E

\( f(x) \in V_0 \Rightarrow f(x) \in V_1 \quad : \quad V_0 \subset V_1 \)
For Haar

1. Scaling fn is \( I \) to its integer translate (only for Haar)

2. \( V_0 \subset V_1 \subset V_2 \subset \ldots \)
   nesting of subspaces.
   if \( f(x) \in V_j \) then \( f(2x) \in V_{j+1} \)

3. Only fn common to all \( V_j \) is \( f(x) = \mathbf{0} \)

\[ V_{-\infty} = \{ \mathbf{0} \} \]
Any function can be represented with arbitrary precision.

\[ V_0 = \{ L^2(\mathbb{R}) \} \]

- Can write \( \Phi_{j,k} \) as a linear combination of \( \Phi_{j+1,k} \)

\[
\Phi_{j,k}(x) = \sum_n C_n \Phi_{j+1,n}(x)
\]

\[
\Phi_{j,k}(x) = \sum_n h_n \phi(n) 2^{j+1} 2^{j+1} \phi(2^j x - n)
\]

Set \( j = k = 0 \) \( \Rightarrow \Phi_{0,0} = \phi(x) \)

\[
\phi(x) = \sum_n h_n \phi(n) \sqrt{2} \phi(2x - n)
\]

\( \phi(x) \) can be built from admissible resolution copies of itself, i.e., from \( \phi(2x) \)

Expansion for \( V_j \) is linear sum of \( V_{j+1} \)

Fig 7.15f 6+10 7
FIGURE 7.11
Some Haar scaling functions.
Wavelet fun

- Span the difference between 2 adjacent subspaces $V_j$ and $V_{j+1}$

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$

$$U_2 = V_0 \oplus W_0$$

$$V_j \oplus W_j$$

- $\psi_{i,k}(x) = 2^{-j/2} \psi(2^{-j} x - k) \quad k \in \mathbb{Z}$

- $W_j = \text{Span} \left\{ \psi_{i,j,k}(x) \right\}$

- $V_{j+1} = V_j \oplus W_j$

- $\text{Union of spaces}$
- Orthogonal complement of \( V_j \) in \( V_{j+1} \) is \( W_j \)

\[
\Rightarrow \langle \phi_{j,k}, \psi_{j,l} \rangle = 0 \quad \forall j, k, l \in \mathbb{Z}
\]

\[
L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \ldots
\]

\[
= V_1 \oplus W_1 \oplus W_2 \oplus \ldots
\]

\[
= \ldots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots
\]

we need to deal with \( \phi \) only \( \psi \).

- If \( f \in V_0 \)

\[
f = \text{linear comb of scaling \( \phi \) in } V_0
\]

\[
+ \text{linear comb of wavelet from } W_0
\]
\[ L^2(\mathbb{R}) = V_j \oplus W_j \oplus W_{j+1} \oplus \ldots \]

\[ = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots \]

\[ = V_5 \oplus W_6 \oplus W_7 \oplus W_8 \oplus \ldots \]

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**Theorem by Burnside**

\[ h\psi(n) = (-1)^n \quad h\phi(1-n) \]

\[ \Rightarrow \text{from } \phi(x) \rightarrow h\phi \rightarrow h\psi \rightarrow \psi \]
For Haar

\[ \psi(x) = \begin{cases} 
1 & 0 \leq x < 0.5 \\
-1 & 0.5 \leq x < 1 \\
0 & \text{otherwise}
\end{cases} \]

Show Fig 7.14 \& w

Wavelet Series Expansion

Arbitrary \( j_0 \)

\[ f(x) = \sum_{k} c_{j_0}(k) \phi_{j_0, k}(x) + \sum_{j=j_0}^{\infty} \sum_{k} d_{j}(k) \psi_{j, k}(x) \]

If \( \phi \) orthonormal on tight frame:

\[ c_{j_0}(k) = \langle f(x), \phi_{j_0, k} \rangle = \int f(x) \phi_{j_0, k}(x) \, dx \]

\[ d_{j}(k) = \langle f(x), \psi_{j, k} \psi (x) \rangle = \int f(x) \psi_{j, k}(x) \, dx \]
FIGURE 7.14
Haar wavelet functions in $W_0$ and $W_1$. 

\( \psi(x) = \psi_{0,0}(x) \)

\( \psi_{0,2}(x) = \psi(x - 2) \)

\( \psi_{1,0}(x) = \sqrt{2} \psi(2x) \)

\( f(x) \in V_1 = V_0 \oplus W_0 \)

\( f_0(x) \in V_0 \)

\( f_0(x) \in W_0 \)
A wavelet series expansion of $y = x^2$ using Haar wavelets.
Show Fig 7.15

Discrete Wavelet Transform

So far dealt with \( f(x) \) \( x \) real.

Now deal with \( f(n) \) \( n \) integer \( \Rightarrow \) sequence not \( f(x) \).

Forward DWT coefficients for \( f(n) \), (assuming tight frame orthogonal)

\[
W_\phi (j_0, k) = \frac{1}{\sqrt{m}} \sum_{n} f(n) \phi_{j_0, k}(n)
\]

\[
W_\psi (j, k) = \frac{1}{\sqrt{m}} \sum_{n} f(n) \psi_{j, k}(n)
\]

\( j \geq j_0 \)

Then

\[
f(n) = \frac{1}{\sqrt{m}} \sum_{k} W_\phi (j_0, k) \phi_{j_0, k}(n) +
\]

\[
\frac{1}{\sqrt{m}} \sum_{j=j_0}^{\infty} \sum_{k} W_\psi (j, k) \psi_{j, k}(n)
\]
Continued Wavelet Transform

- Already discussed last time
\[
W_\psi(s, \tau) = \int_{-\infty}^{\infty} f(x) \psi_{s, \tau}(x) \, dx
\]

\[
\psi_{s, \tau}(x) = \frac{1}{s} \psi\left(\frac{x-\tau}{s}\right)
\]

\[s = \text{Scale} \quad \tau = \text{Translation}\]

Inverse Wavelet
\[
f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_\psi(s, \tau) \psi_{s, \tau}(x) \, ds \, d\tau
\]

where
\[
C_\psi = \int_{-\infty}^{\infty} \left|\hat{\psi}(\mu)\right|^2 \, d\mu
\]

- Compression \( 0 < s < 1 \)
- Dilation \( s > 1 \)

- Show Fig 7.16
FIGURE 7.16
The continuous wavelet transform (c and d) and Fourier spectrum (b) of a continuous 1-D function (a).
Fast Wavelet Transform

\[ \Phi(x) = \sum_n h \Phi(n) \sqrt{2} \Phi(2x - n) \]

\[ x \leftarrow \frac{j}{2} x - k \]

\[ \Phi(2^j x - k) = \sum_n h \Phi(n) \sqrt{2} \Phi(2^j (2x - k) - n) \]

\[ = \sum_m h \Phi(m - 2k) \sqrt{2} \Phi(2^{j+1} x - m) \]

Similarly

\[ \Psi(2^j x - k) = \sum_m h \Psi(m - 2k) \sqrt{2} \Phi(2^{j+1} x - m) \]

Recall

\[ d_j(k) = \int f(x) \frac{1}{2^j} \Psi(2^j x - k) \, dx \]

\[ d_j(k) = \sum_m h \Psi(m - 2k) c_{j+1}(m) \]

Similarly

\[ c_j(k) = \sum_m h \Phi(m - 2k) c_{j+1}(m) \]
\[ C_j(k) \rightarrow W\psi(j_k) \{ \text{DWT} \}
\]

\[ d_j(k) \rightarrow W\phi(j_k) \]

as \[ f(x) \rightarrow f(2^m) \]

Then

\[ W\psi(j_k) = \sum_{m} h\psi(m-2k) \left[ W\phi(j+1)_m \right] \]

\[ W\phi(j_k) = \sum_{m} h\phi(m-2k) W\phi(j+1)_m \]

\[ \Rightarrow \begin{cases} 
W\psi(j, k) = h\psi(-n) * W\phi(j+1)_n \\
W\phi(j, k) = h\phi(-n) * W\phi(j+1)_n 
\end{cases} \]

\[ h\psi(-n) \quad \text{and} \quad h\phi(-n) \]

Subband Analysis

Since Convolution

Can use FFT or other fast algorithm. See Fig. 7.18
FIGURE 7.10
(a) A two-stage or two-scale FWT analysis bank and (b) its frequency splitting characteristics.