Assignment 3 Solutions

1. Nishimura 3.3

Answer

\[ G_r = \frac{dB_z}{dr} \quad r = \sqrt{x^2 + y^2} \geq 0 \quad \text{all circularly symmetric objects } m(r) \]

a) We are told that the gradient \( \frac{dB_z}{dr} \) is radial.

The field at a particular \( r \) distance away will be

\[ G_r r \] (on a concentric ring of radius \( r \))

We can write our signal equation but now using radial coordinates to integrate over all space (remember extra \( r \) term when converting to radial coordinates)

\[ S(t) = \int_0^r \int_0^{2\pi} m(r) e^{-i\gamma \int_0^t G_r(r) \delta t} \cdot r \, dr \, d\theta \]

\[ \text{Remember that the gradient is a function of time which means we must integrate,} \]

\[ S(t) = \int_0^r m(r) e^{-i\gamma \int_0^t G_r(r) \delta t} \cdot r \, dr \int_0^{2\pi} d\theta \]

\[ S(t) = \int_0^r m(r) e^{-i\gamma \int_0^t G_r(r) \delta t} \cdot r \, dr \int_0^{2\pi} d\theta \]

\[ = 2\pi \int_0^r m(r) e^{-i\gamma \int_0^t G_r(r) \delta t} \cdot r \, dr \]

\[ = 2\pi \mathcal{F} \mathcal{F} \int_0^r m(r) \frac{d}{dr} \mathcal{F} = \gamma \int_0^t G_r(r) \, dr \]

\[ k_r(t) = \gamma \int_0^t G_r(r) \, dr \]
b) \[ S(r-R) \quad G_r(t) = G_r \quad t > 0 \]

\( S(r-R) \) is an impulse that only has value when \( r = R \)

From part a) our baseband signal is

\[
\begin{align*}
\frac{\partial}{\partial t} \int_{-\infty}^{\infty} S(r-R) \cdot I \mathfrak{F} k_r(t) &= \mathfrak{F} \int_{0}^{\infty} G_r(t) \, dt \\
&= 2\pi \int_{0}^{\infty} S(r-R) e^{-j\gamma \left( \int_{0}^{t} G_r \, dt \right)} \cdot r \, dr \\
&= 2\pi \int_{0}^{\infty} S(r-R) e^{-j\gamma \mathfrak{F} G_r(t) \cdot \sqrt{r}} \, dr \\
&= \frac{2\pi \mathbf{R} e^{-j\gamma G_r(t) \cdot R}}{\sqrt{r}} \\
\text{note: } \int S(x-x_0) f(x) \, dx = f(x_0) \quad \text{(sifting property)}
\end{align*}
\]

\[
\begin{align*}
\text{c) } & \quad \text{circular pillbox of unit amplitude.} \\
& \quad \text{radius } r = \frac{D}{2} \\
\text{We can represent this with a circ function.} \\
\text{circ}(r) \text{ is a circular function with unit diameter } (r = \frac{D}{2}) \\
\text{To get our pillbox of diameter } D \Rightarrow m(r) = \text{circ}(\frac{D}{2}) \\
\text{From part a) the baseband signal is} \\
\begin{align*}
S(t) &= 2\pi \int_{-\infty}^{\infty} \text{circ}(\frac{D}{2}) \cdot \mathcal{F}^{-1} S(t) \, dt \int_{0}^{\infty} \mathfrak{F} G_r(t) \cdot \sqrt{r} \, dr
\end{align*}
\]
\[ S(f) = \frac{\gamma^2}{\pi^2} s(t) \frac{\gamma}{2} = \frac{\gamma^2}{\pi} \text{arc}(\gamma D) \frac{\gamma}{2} \]

\[ = \frac{\gamma}{\pi} \text{circ}(\gamma D) \]

\[ \text{slope} = \frac{\pi D}{D/2} = \frac{2\pi D}{\gamma} = \frac{\gamma}{\pi} \]

**Given** that \( S(f) = \frac{\gamma}{\pi} \text{circ}(\gamma D) \) \( r \) and we want to recover our original \( m(r) = \text{circ}(\gamma D) \) divide by \( \frac{\gamma}{\pi} r \) for \( 0 < r < D/2 \)

\[ \frac{S(f)}{\frac{\gamma}{\pi} r} \Rightarrow \frac{\frac{\gamma}{\pi} \text{circ}(\gamma D)}{\frac{\gamma}{\pi} r} \Rightarrow \text{circ}(\gamma D) \]

\( 0 < r < D/2 \)

d.) \[ BW = \gamma \frac{G_r}{d} = \frac{\gamma}{\frac{\pi}{a}} \text{cm} = \left( \frac{\gamma}{\frac{\pi}{a}} \right) \left( \frac{2.62}{\text{cm}} \right) \left( 10 \text{ cm} \right) \]

\[ = \left( \frac{4.3576 \text{ kHz}}{\frac{\pi}{a}} \right) \left( \frac{2.62}{\text{cm}} \right) \left( 5 \text{ cm} \right) = 4.2576 \text{ kHz} \]
2. Precession in Tops (Double Points: brain teasers)

(a) Explain why only spinning tops precess. That is, explain why static tops will not.

Answer

Let's first look at a static top.

If the top is not spinning, the force of gravity will make it fall.

\[
\tau = mg
\]

\(m\) = mass of one
\(r\) = distance between cone's center of mass and the pivot point (origin)

However, if a top is spinning, the torque makes it precess.

\[
\tau = I \omega
\]

The top's angular momentum is primarily directed along its axis.

\[
\vec{\tau} = \vec{I} \times F_{\text{gravity}} \quad \text{(direction into board)}
\]

We know the \(\tau = \frac{dl}{dt}\)

\[
\vec{\tau} = \vec{I} \times F_{\text{gravity}} \Rightarrow \text{precession}
\]

In MKS, the gravitational field \(\sim\) magnetic field \((F_{\text{gravity}} \sim B)\)

angular momentum \(\sim\) magnetic moment \((\vec{I} \sim \vec{\mu})\)

\[
\vec{A} = \gamma \vec{\sigma} = \gamma (\vec{\mu} I)
\]

\[
\vec{\sigma} = \vec{\mu} \times B
\]

spin angular momentum \(\frac{d}{dt} \vec{\sigma} = \vec{\tau} = \vec{I} \times F_{\text{gravity}}\)

\[
\frac{d}{dt} \vec{\sigma} = \vec{I} \times B
\]

Black equation

(b) Suppose you wanted to create an approximate macro version of the precessing quantum nuclear magnetic moment. Dr. Bore thinks you could do this by embedding a permanent magnetic (PM) moment within a toy top. Would you embed the PM perpendicular or parallel to the
rotation axis? If you altered the applied external magnetic field, do you believe you would alter the mechanical precession frequency of the top— that is, the precession would change from that determined merely by gravity? Could you create both positive and negative gyromagnetic ratios?

To approximate a spin, the magnetic moment must be pointing at the same direction of the angular momentum. Therefore the magnet should be aligned with the top. Altering an external field can either contribute or subtract from the gravitational force. The magnetic field can be altered to contradict the effect of gravity, thus stopping the precession. If the magnetic field becomes a dominating factor, precession will change direction, effectively alternating the gyromagnetic ratio.
3. Magnetic field steps

Suppose you placed two test tubes (each 50 mL) at 1.5 T and one test tube (50 mL) at 1.6 T. Suppose both coils were visible with equal sensitivity to the RF coil. Sketch the intensity of the received signal as a function of frequency.

Answer

\[ \begin{align*}
\text{A:} & \quad 50 \text{ mL x 2} \quad \text{at} \quad 1.5 \text{ T} \\
\text{B:} & \quad 50 \text{ mL x 2} \quad \text{at} \quad 1.6 \text{ T}
\end{align*} \]

We need to figure out at what frequency the H\(^+\) protons in these tubes will be precessing at (i.e. the resonance frequency).

Using \( f = \gamma B_z \) (where \( \gamma = \frac{\hbar}{2m} \))

\[ \begin{align*}
f_A &= \frac{42.576 \text{ MHz}}{\gamma} \times 1.5 \gamma = 63.9 \text{ MHz} \\
f_B &= \frac{42.576 \text{ MHz}}{\gamma} \times 1.6 \gamma = 68.1 \text{ MHz}
\end{align*} \]

Plotting intensity versus frequency:

\[ \begin{array}{c}
|S| \\
\hline
f_A & f_B & f [\text{MHz}]
\end{array} \]

**Note:** that the 2 tubes at 1.5 T will lead to higher signal intensity at \( f_A \) than at \( f_B \).
4. RF Field

(a) Find the amplitude of an RF pulse that performs a 90 degree excitation in exactly 1 ms at 1.5T.

(b) Find the amplitude of an RF pulse that performs a 90 degree excitation in exactly 1 ms at 3T.

Answer

a) $90^\circ$ excitation $\Rightarrow \gamma/2$, flip angle

\[
\gamma = 1\text{ ms}
\]

\[
\gamma = \frac{\gamma}{\text{MHz}} \times \frac{1}{\text{ms}} \times \frac{\pi}{\gamma} \text{ (2\pi radians)}
\]

\[
\gamma/2 = \gamma B_1 \gamma
\]

\[
\gamma/2 = \frac{42.576\text{ MHz}}{\text{T}} B_1 (1\text{ ms}) \times \frac{\pi}{\gamma} \text{ radians}
\]

\[
\gamma/2 = \frac{42.576 \times 10^{-6} \text{ Hz}}{\text{T}} B_1 (1 \times 10^{-3}) \text{ radians}
\]

\[
\gamma/4 = \frac{42.576 \times 10^{-6} \text{ Hz}}{\text{T}} \left( \frac{1}{10^3 \text{ seconds}} \right) B_1
\]

\[
\gamma/4 = \frac{42.576 \times 10^{-6}}{\text{T}} B_1
\]

\[
B_1 = \frac{\gamma/4}{42.576 \times 10^{-3}} = 5.8719 \times 10^{-6} \text{ T} \approx 6 \gamma T
\]

b) No the amplitude will not change at 3T. The base frequency of $B_1(t)$ will change since base frequency is equal to $\gamma B_0$, but the Amplitude does not change.
5. Gradient field bandwidth

Sketch the spectrum of received signal on a 1.5 T MRI scanner versus frequency for an object of diameter = 25 cm with a gradient field of peak amplitude 4 G/cm.

Answer

\[ B_0 = 1.5 \, T \]
\[ G = 40 \, \text{mT/m} \]

The center of our sample will be precessing at a frequency dependent on the field strength:

\[ f_{\text{center}} = y \times 1.5 \, T = 42.576 \, \text{MHz} \left( \frac{1.5 \, T}{y} \right) = 63.9 \, \text{MHz} \]

The edge of the samples will be precessing at a frequency dependent on both the field strength and the peak amplitude of the gradient:

\[ 12.5 \, \text{cm} \times \frac{y}{100 \, \text{cm}} \times \frac{40 \, \text{mT}}{y} \times \frac{1}{1000 \, \text{mT}} = 0.005 \, T = 5 \times 10^{-3} \, T \]

\[ 5 \times 10^{-3} \, T \left( \frac{y}{8} \right) = 5 \times 10^{-3} \left( 42.576 \, \text{MHz/T} \right) = 0.213 \, \text{MHz} \]

\[ f = \left[ \text{MHz} \right] \]

\[ 63.9 - 0.213 = 63.7 \, \text{MHz} \]
\[ 63.9 + 0.213 = 64.1 \, \text{MHz} \]
6. **RF Excitation.** If we apply an RF waveform in addition to a gradient G, we can excite a slice through an object. Assume that the RF envelope $B_1(t)$ is the 6 ms segment of a sinc(), as shown

$$B_1(t) = \text{sinc}(2(t - 3))\text{rect}((t - 3)/6)$$

where times are in ms. The Fourier transform is

$$\mathcal{F}\{B_1(t)\} = e^{-j2\pi f} \left[ \frac{1}{2} \text{rect}(f/2) * 6 \text{sinc}(6f) \right]$$

The phase term doesn’t effect the slice width, so we’ll ignore it. The Fourier transform is a 2 kHz wide rect convolved with a 1/6 kHz wide sinc. The result will be about 2 kHz wide. This is illustrated below, where the normalized functions are plotted,

To find the width of the slice this pulse excites, first note that a gradient of 0.94 G/cm produces a linear frequency variation of

$$\frac{\gamma}{2\pi} G = (4.257 \text{ kHz/G})(0.94 \text{ G}) = 4 \text{ kHz/cm}$$

The RF pulse has a spectrum that is 2 kHz wide, so the excited slice will be

$$2 \text{ kHz}/(4 \text{ kHz/cm}) = 0.5 \text{ cm}$$

in width. This is a typical slice width.
7 **Non-Linear gradients.** One of the key elements in MRI is the use of a gradient field $G$ which establishes a linear relationship between resonance frequency and position. While the linear model is convenient for analysis, real gradients are seldom exactly linear. In this problem we will look at some of the consequences of gradient non-linearity. This is a REAL situation in every scanner!

(a) Consider a gradient system with the response shown by the solid line. A linear model is shown as the dashed line.

Assume the object is a sequence of rectangles of uniform intensity.

Sketch the one dimensional image we would get if we encode using the real gradient (solid line) but use the linear approximation (dashed line) when we reconstruct the data (i.e. assign spatial positions to different frequencies.) Things to look for are spatial distortion, and intensity variations.

**Answer** We can solve this graphically. First, the non-linear gradient encodes position as frequency. If we can use the plotted curve to figure out the frequency that corresponds to the edges of each of the rectangles.
Then we decode each of these frequencies, assuming that the gradients were the ideal linear gradients. Here we go from the actual frequency generated by the non-linear gradient, and map that back to space using the linear gradient assumption:

What happens is that as the slope of the gradient waveform falls off, the image becomes more compressed. At the edges of the image the rectangles are much narrower. However, each has to have the same area (there is the same number of water molecules producing signal) so that the amplitude has to go up inversely with the width.

(b) The non-linearity in the gradient can be measured, and then used to more accurately reconstruct the data. Assume that the gradient field produces a frequency

$$\omega(z) = \gamma(G_{\text{ideal}}z + Hz^3)$$

where $G_{\text{ideal}} = 0.235 \text{ G/cm}$, and $H = -1.9 \times 10^{-4} \text{ G/(cm}^3\text{)}$, and $z$ ranges from ±20 cm. If we have a data acquisition window of 10 ms, we can resolve frequencies of 100 Hz. What spatial resolution does this provide at $z = 0$, 10, and 20 cm?
Answer The question is, what spatial distance corresponds to 100 Hz at these different positions. The slope of the non-linear gradient is

\[ \frac{d\omega}{dz} = \gamma(G_{\text{ideal}} + 3Hz^2) \]

This means that

\[ \frac{dz}{d\omega} = \frac{1}{\gamma(G_{\text{ideal}} + 3Hz^2)} \]

The approximate spatial distance that corresponds to 100 Hz is

\[ \Delta z = \Delta \omega \cdot \frac{dz}{d\omega} = \frac{2\pi \times 100}{\gamma(G_{\text{ideal}} + 3Hz^2)} \]

Substituting in for \( G_{\text{ideal}}, H \) and \( z \), we get \( \Delta z \) is 1 mm at \( z = 0 \), 1.32 mm at \( z = 10 \) cm, and 33.6 mm at \( z = 20 \) cm.

(c) Would this gradient profile work for spatial encoding for MRI? Why or why not? Assume that the object extends from -20 cm to 20 cm.

Answer Since the non-linear gradient is not monotonic, two spatial locations encode to the same frequency. Then we can’t tell where the signal came from by looking at its spectrum. We can’t use this non-linear gradient to unambiguously encode spatial information, so we can’t use it for imaging, at least not over the entire FOV. It would work fine if the object were limited, say to +/- 10 cm.
8. Design of Time-Optimal Gradient Waveforms

A key element in pulse sequence design in MRI is the design of the gradient waveforms. A very common problem is to design a gradient waveform with a certain desired area. To minimize the duration of the sequence, we often would like the waveform to be as short as possible. However, the gradient pulse has to be realizable by the system and therefore must satisfy the system constraints of maximum gradient amplitude and slew-rate.

Solution: A minimum time solution is always either limited by the slew-rate or by the maximum gradient amplitude. There are two cases we should consider depending on the desired gradient area:

When the desired area is smaller than \( \frac{G_{\text{max}}^2}{S_{\text{max}}} \) then the gradient magnitude is never maxed out, so the solution is a triangle. For an area larger than \( \frac{G_{\text{max}}^2}{S_{\text{max}}} \) the solution is a trapezoid.

For a triangle we define \( t_1 = \sqrt{\frac{\text{area}}{S_{\text{max}}}} \) and get:

\[
G(t) = \begin{cases} 
S_{\text{max}}t, & 0 \leq t \leq t_1 \\
\frac{2\sqrt{\text{area}}}{S_{\text{max}}} - S_{\text{max}}t, & t_1 \leq t \leq 2t_1 
\end{cases}
\]

For a trapezoid we define \( t_1 = \frac{G_{\text{max}}}{S_{\text{max}}} \), \( t_2 = \frac{\text{area}}{G_{\text{max}}} \) and \( t_3 = \frac{\text{area}}{G_{\text{max}}} + \frac{G_{\text{max}}}{S_{\text{max}}} \) and get,

\[
G(t) = \begin{cases} 
S_{\text{max}}t, & 0 \leq t \leq t_1 \\
G_{\text{max}}, & t_1 \leq t \leq t_2 \\
\left( \frac{\text{area}}{G_{\text{max}}} + \frac{G_{\text{max}}}{S_{\text{max}}} \right) S_{\text{max}} - S_{\text{max}}t, & t_2 \leq t \leq t_3 
\end{cases}
\]

Given that the system is limited to maximum gradient amplitude \( G_{\text{max}} = 4G/cm \) and a slew-rate of \( S_{\text{max}} = \frac{dG(t)}{dt} = 15000G/cm/s \)

(a) Find the shortest gradient waveform that has an area of \( \int G(\tau)d\tau = 8e-4 \ G*s/cm \). Draw the waveform. Point out the maximum gradient, and its duration. What is the shape of the waveform?
Solution:
For the parameters we get that the desired area is smaller than \( \frac{G_{\text{max}}^2}{S_{\text{max}}} = 10.67 \times 10^{-4} \), so the solution is a triangle. The maximum gradient is \( G = \sqrt{\text{area}} \frac{S_{\text{max}}}{2} = 3.464 \text{G/cm} \) and the duration is \( T = 2\sqrt{\text{area}} \frac{S_{\text{max}}}{2} = 0.462 \text{ms} \).

(b) Find the shortest gradient waveform that has an area of \( \int_{\tau} G(\tau)d\tau = 16 \times 10^{-4} \text{G*s/cm} \). Draw the waveform. Point out the maximum gradient, and its duration. What is the shape of the waveform?

Solution:
Now, the desired area is bigger than \( \frac{G_{\text{max}}^2}{S_{\text{max}}} = 10.67 \times 10^{-4} \), so the waveform is a trapezoid with maximum gradient of 4G/cm and a duration of \( T = \frac{\text{area}}{G_{\text{max}}} + \frac{G_{\text{max}}}{S_{\text{max}}} = 0.6667 \text{ms} \).

(c) Matlab assignment: In this part, we will write a matlab function to design minimum-time gradient waveforms. This function will be used later in class, so make sure you get it right. Write a function that accepts the desired gradient area (in G*s/cm), the maximum gradient amplitude (in G/cm), the maximum slew-rate (in G/cm/s) and sampling interval (in s). The function will return (a discrete) shortest gradient waveforms that satisfy the constraints:

\[
g = \text{minTimeGradientArea}(\text{area}, \text{Gmax}, \text{Smax}, \text{dt});
\]

Solution: There are many ways to implement this. Here’s an example of sampling the continuous function of the analytic gradient waveform solution.

function g = minTimeGradientArea(area, Gmax, Smax, dt)

% g = minTimeGradientArea(area, Gmax, Smax, dt)

A_triang = Gmax^2/Smax;

if area <= A_triang
    disp('Triangle')
    t1 = sqrt(area/Smax);
    T = 2*t1;
    N = floor(T/dt);
    t = [1:N]'*dt;

    idx1 = find(t < t1);
    idx2 = find(t >= t1);

    g = zeros(N,1);
    g(idx1) = Smax*t(idx1);
    g(idx2) = 2*sqrt(area*Smax)-Smax*t(idx2);
else

end
disp('Trapezoid')
t1 = Gmax/Smax;
t2 = area/Gmax;
t3 = area/Gmax + Gmax/Smax;

T = t3;
N = floor(T/dt);
t = [1:N]'*dt;

idx1 = find(t < t1);
idx2 = find((t>=t1) & (t < t2));
idx3 = find(t>=t2);

g = zeros(N,1);
g(idx1) = Smax*t(idx1);
g(idx2) = Gmax;
g(idx3) = (area/Gmax + Gmax/Smax)*Smax - Smax*t(idx3);
end

disp(sprintf('Maximum gradient:%fG/cm
Duration:%fms
',max(g(:)),t(end)*1000));

• Plot the result of the function for area=6e-4, Gmax=4, Smax = 15000, dt=4e-6;
• Plot the result of the function for area=6e-4, Gmax=1, Smax = 5000, dt=4e-6;
  Solution: As seen below, gradients with higher slew-rate and maximum amplitude can produce the same gradient area faster.