EE 225E / BIOE 265 LECTURE 3 01/23/12

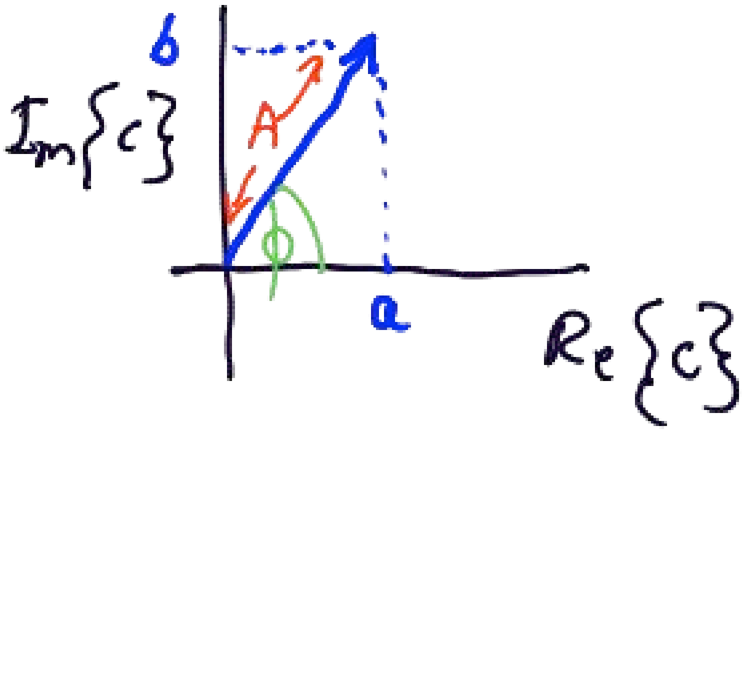
Preliminaries:

complex numbers:

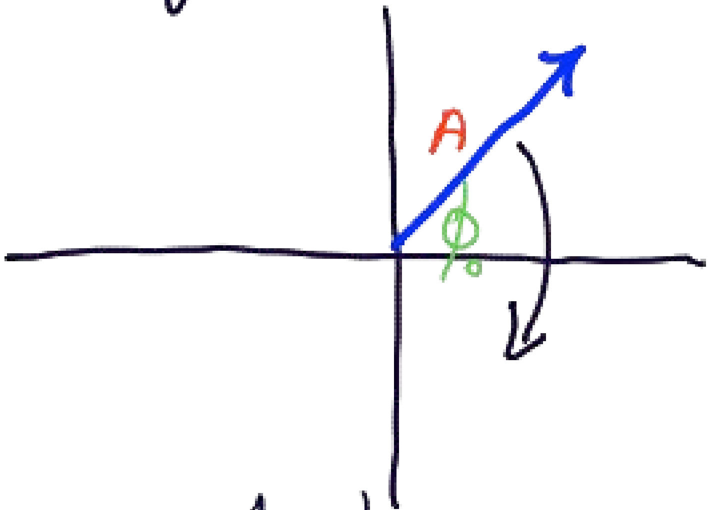
$$c = a + ib \qquad i = \sqrt{-1}$$

or phasors:

$$c = Ae^{i\phi} = A(\cos\phi + i\sin\phi)$$



It is convinient to use complex # to describe ②
a vector in a plane



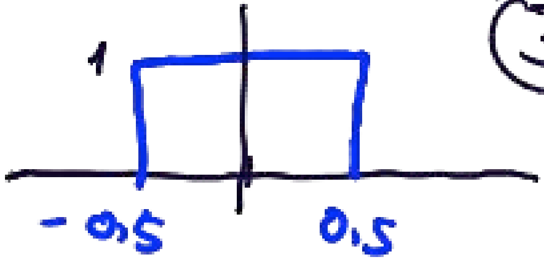
$c(t) = Ae^{-i\omega t + i\phi_0}$ ⇒ clock-wise rotation w/ angular freq. ω

VERY USEFUL TO DESCRIBE

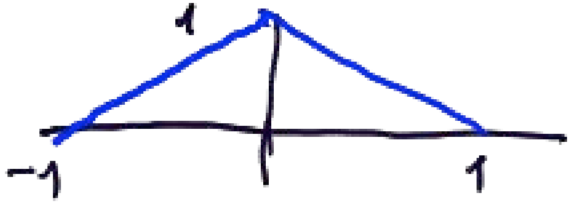
(transverse) MAGNETIZATION

USEFUL FUNCTIONS:

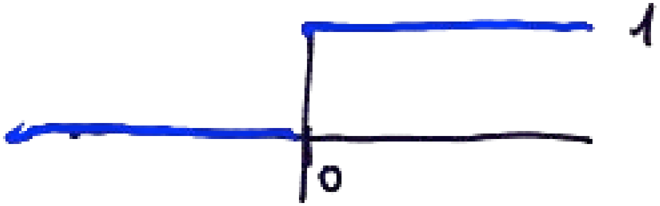
$$\Pi(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



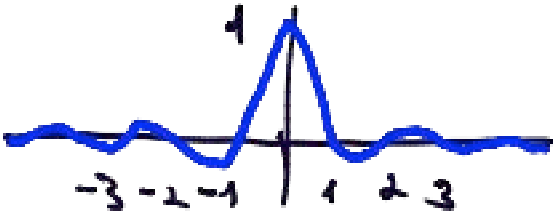
$$\Lambda(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$u(x) = \begin{cases} 1 & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

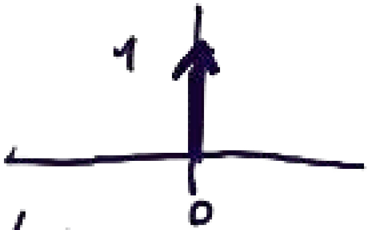


$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



The Impulse function $\delta(x)$

DEFINITION $\Rightarrow$ $\displaystyle\int_{-\infty}^{\infty} \varphi(x)\delta(x)\,dx = \varphi(0)$



Properties:

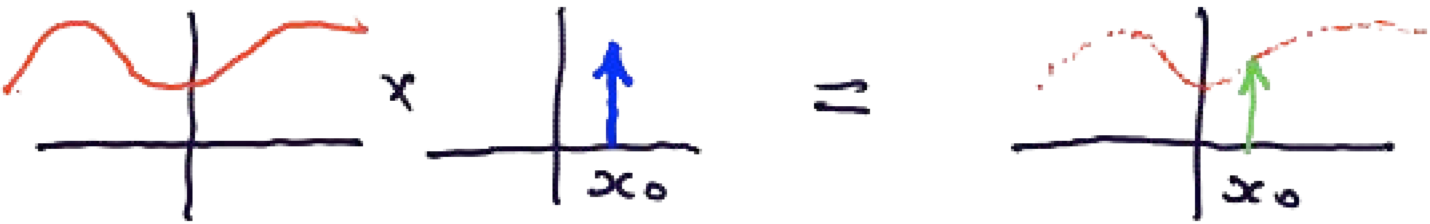
(1) $\delta(x) = 0 \quad \forall x \neq 0$

(2) $\delta(x) \to \infty \quad | \; x = 0$

(3) $\displaystyle\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

(4) $\delta(ax) = \dfrac{1}{|a|}\delta(x)$

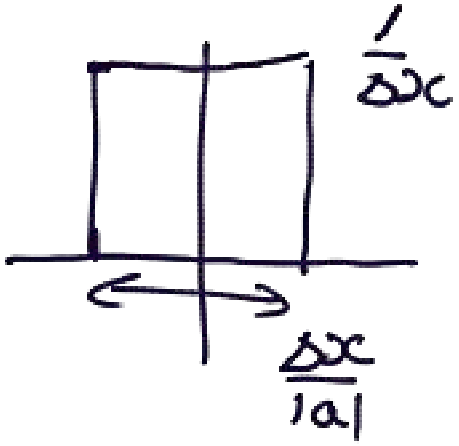
(5) $\varphi(x)\delta(x-x_0) = \varphi(x_0)\delta(x-x_0)$



many realizations to $\delta(x)$:

example: $$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Pi\left(\frac{x}{\Delta x}\right)$$

$$\delta(ax) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Pi\left(\frac{ax}{\Delta x}\right) =$$



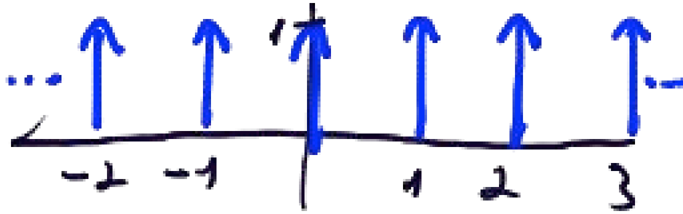
$$\text{area} = \frac{\Delta x}{|a|} \cdot \frac{1}{\Delta x} = \frac{1}{|a|}$$

$$= \frac{1}{|a|} \delta(x)$$

Impulse Train $Ш(x)$

(impulse train
shah function
comb)

$$Ш(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



what about:

$$\sum_{n=-\infty}^{\infty} \delta(x - \Delta x n) = \sum \delta\left(\Delta x\left(\frac{x}{\Delta x} - n\right)\right) =$$

$\delta(ax) = \frac{1}{|a|}\delta(x)$

$$= \frac{1}{\Delta x} \sum \delta\left(\frac{x}{\Delta x} - n\right) = \frac{1}{\Delta x} Ш\left(\frac{x}{\Delta x}\right)$$

USEFUL TO MODEL SAMPLING!

SYSTEMS

$$g(x) \longrightarrow \boxed{T\{\,\}} \longrightarrow y(x)$$

Properties:

Linear

$$T\{g_1(x)+g_2(x)\} = y_1(x)+y_2(x)$$

$$T\{a\,g(x)\} = a\,y(x)$$

shift-invariant

$$T\{g(x-x_0)\} = y(x-x_0)$$

convolution: important math. operator

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(x-\tau)\,h(\tau)\,d\tau$$

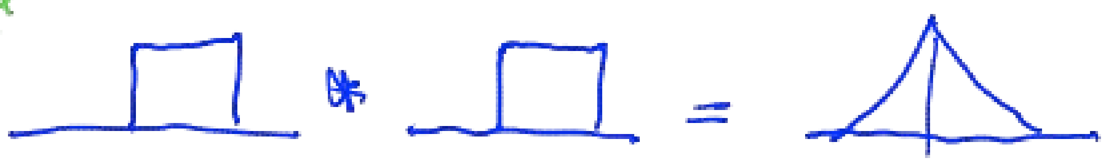
why useful??

$$\delta(x) \longrightarrow \boxed{LSI} \longrightarrow h(x)$$ Impulse response

$$g(x) \longrightarrow \boxed{LSI} \longrightarrow g(x) * h(x)$$ for ANY input $g(x)$!

convolution $\Longrightarrow$ blur

Example



$$\Pi(x) * \Pi(x) = \Lambda(x)$$

$\boxed{\text{show example}}$

FOURIER TRANSFORM (1D)

complex val.

$$F(k_x) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi k_x x}\, dx$$

frequency spectrum

How come?

inverse xform:

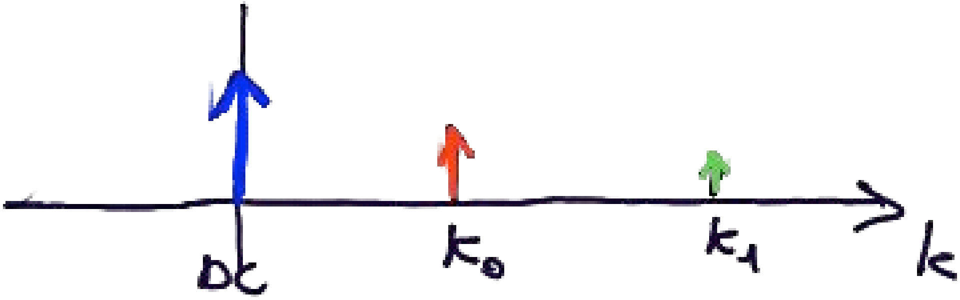
$$f(x) = \int_{-\infty}^{\infty} F(k_x)\, e^{+i2\pi k_x x}\, dk_x$$

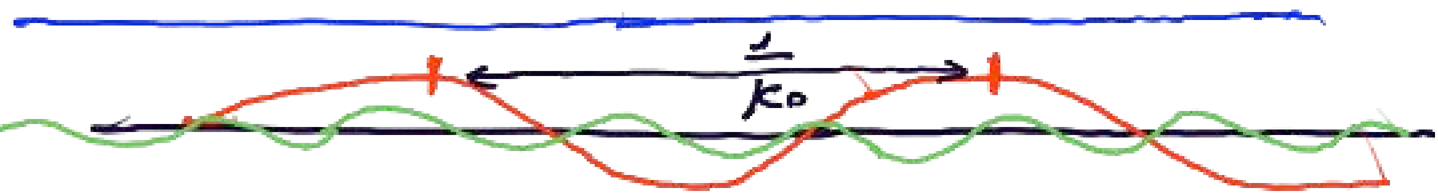
Units: If x in cm $\Rightarrow$ k is $1/\text{cm}$

t in s $\Rightarrow$ f is Hz

Intuition for FT

$F(k_0)$ IS THE ENERGY OF $f(x)$ at freq. k_0





(-) Amplitude $\Longleftrightarrow$ Height

(-) PHASE $\Longleftrightarrow$ shift

Some FT

1. $\Pi(x) \Rightarrow \mathrm{sinc}(k)$

2. $1 \Rightarrow \delta(k)$

   $e^{i2\pi k_0 x} \Rightarrow \delta(k-k_0)$

   } Periodic $\Rightarrow$ impulsive

3. $\delta(x-x_0) \Rightarrow e^{i2\pi x_0 k}$ impulsive $\Rightarrow$ periodic

4. $\text{Ш}(x) \Rightarrow \text{Ш}(k)$

- self transform

  impulsive & Periodic $\Rightarrow$ impulsive & Periodic

Properties of Fourier Transform

$$f(x) \Rightarrow F(k)$$

scaling

(1) $f(ax) \Rightarrow \frac{1}{|a|} F\left(\frac{k}{a}\right)$

shrink $\Leftrightarrow$ expand
expand $\Leftrightarrow$ shrink

shift

(2) $f(x-x_0) \Rightarrow e^{-i2\pi x_0 k} F(k)$

shift $\Leftrightarrow$ modulation

modulation

(3) $e^{-i2\pi k_0 x} f(x) \Rightarrow F(k-k_0)$

symmetry

(4) real + even $\Rightarrow$ real + even
real + odd $\Rightarrow$ imag + odd

$f(x)$ real $\Rightarrow F(-k) = F^*(k)$

what is it good for?

convolution

5 $f(x) * g(x) \Longrightarrow F(k) \cdot G(k)$

$f(x) \cdot g(x) \Longrightarrow F(k) * G(k)$

why useful?

$$\mathcal{F}\{f(x) * g(x)\} = F(k)G(k)$$

hard

easy

why Powerful:

$$g(x) \longrightarrow \boxed{h(x)} \longrightarrow y(x)$$

often, want specific output so need to design $g(x) \Rightarrow$ "Deconvolution"

easy with FT

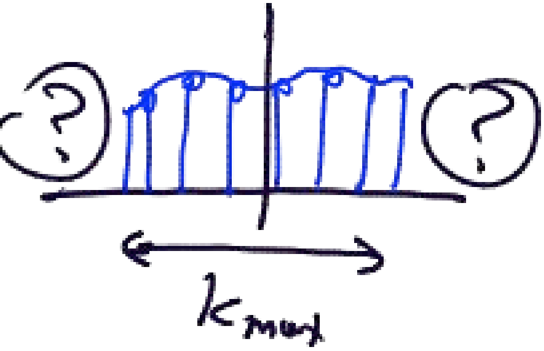
$$Y(k) = G(k)H(k)$$

$$G(k) = \frac{Y(k)}{H(k)}$$

what is the problem?

EXAMPLE:

suppose $F(k)$ known over limited range



Probelms: ① support

② sampling → later

how do we get back $f(x)$?

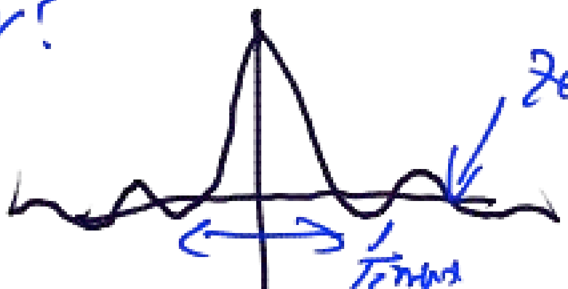
① we know $F(k)\,\Pi\left(\frac{k}{k_{max}}\right)$

FT^{-1}

$$\Rightarrow \quad f(x) * k_{max}\,\mathrm{sinc}(k_{max}x)$$ blurr!

(Q) can you deblurr?

zeros!!!



Less blurr $k_{max} \uparrow$