EE225E/BIOE265 LECTURE 3 01/23/12

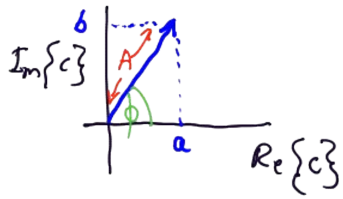
Preliminaries:

complex numbers:

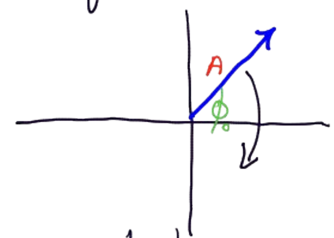
$$c = a + ib \qquad j = \sqrt{-1}$$

or phasors:

$$c = Ae^{i\phi} = A(\cos\phi + i\sin\phi)$$



It is convinient to use complex # to describe ② a vector in a plane

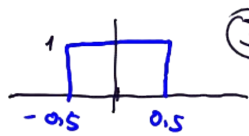


$$c(t) = Ae^{-i\omega t + i\phi_0} \Rightarrow \text{clock-wise rotation w/ angular freq. } \omega$$

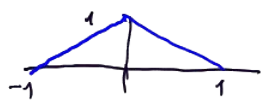
VERY USEFUL TO DESCRIBE (transverse) MAGNETIZATION

USEFUL FUNCTIONS: ③

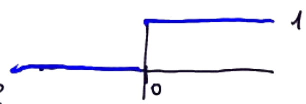
$$\Pi(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



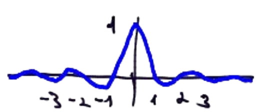
$$\Lambda(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$u(x) = \begin{cases} 1 & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



The Impulse function $\delta(x)$ ④

DEFINITION $\Rightarrow \int_{-\infty}^{\infty} \varphi(x)\delta(x)\,dx = \varphi(0)$

Properties:

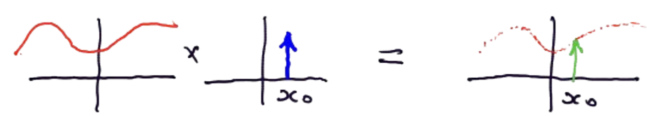
(1) $\delta(x) = 0 \quad \forall x \ne 0$

(2) $\delta(x) \to \infty \quad | \, x = 0$

(3) $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

(4) $\delta(ax) = \frac{1}{|a|}\delta(x)$

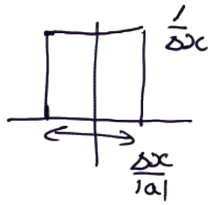
(5) $\varphi(x)\delta(x - x_0) = \varphi(x_0)\delta(x - x_0)$



many realizations to $\delta(x)$: (5)

example: $\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\Pi\left(\frac{x}{\Delta x}\right)$

$\delta(ax) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\Pi\left(\frac{ax}{\Delta x}\right) =$



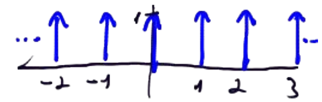
$\text{area} = \frac{\Delta x}{|a|} \cdot \frac{1}{\Delta x} = \frac{1}{|a|}$

$= \frac{1}{|a|}\delta(x)$

Impulse Train $Ш(x)$ (6)

(impulse train
shah function)
comb

$Ш(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$



what about:

$\sum_{n=-\infty}^{\infty} \delta(x - \Delta x n) = \sum \delta\left(\Delta x\left(\frac{x}{\Delta x} - n\right)\right) =$

$\delta(ax) = \frac{1}{|a|}\delta(x)$

$= \frac{1}{\Delta x}\sum \delta\left(\frac{x}{\Delta x} - n\right) = \frac{1}{\Delta x} Ш\left(\frac{x}{\Delta x}\right)$

USEFUL TO MODEL SAMPLING!

SYSTEMS (7)

$g(x) \longrightarrow \boxed{T\{\}} \longrightarrow y(x)$

Properties:

Linear

$T\{g_1(x) + g_2(x)\} = y_1(x) + y_2(x)$

$T\{a\,g(x)\} = a\,y(x)$

shift-invariant

$T\{g(x - x_0)\} = y(x - x_0)$

Convolution: important math. operator (8)

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(x-\tau)\,h(\tau)\,d\tau$$

why useful??

$\delta(x) \longrightarrow \boxed{LSI} \longrightarrow h(x)$ Impulse response

$g(x) \longrightarrow \boxed{LSI} \longrightarrow g(x) * h(x)$ for ANY input $g(x)$!

convolution $\Longrightarrow$ blurr

Example

$\Pi(x) * \Pi(x) = \Lambda(x)$

Show example

FOURIER TRANSFORM (1D)

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complex val. $\rightarrow F(k_x) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi k_x x}\, dx$

frequency spectrum

How come?

inverse xform:

$$f(x) = \int_{-\infty}^{\infty} F(k_x)\, e^{+i2\pi k_x x}\, dk_x$$

Units: If x in cm $\Rightarrow$ k is 1/cm

t in s $\Rightarrow$ f is Hz

Intuition for FT

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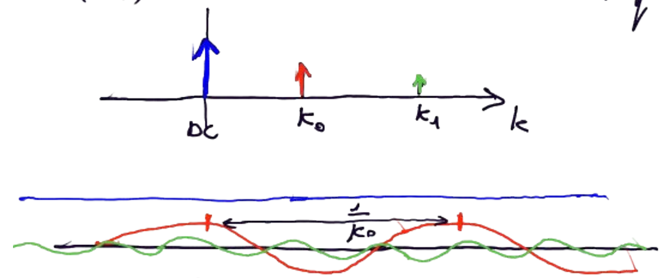
$F(k_0)$ IS THE ENERGY OF $f(x)$ at freq. k_0



(-) Amplitude $\Leftrightarrow$ Height

(-) PHASE $\Leftrightarrow$ shift

Some FT

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① $\Pi(x) \Rightarrow \mathrm{sinc}(k)$

② $1 \Rightarrow \delta(k)$

$e^{i2\pi k_0 x} \Rightarrow \delta(k-k_0)$ } Periodic $\Rightarrow$ impulsive

③ $\delta(x-x_0) \Rightarrow e^{i2\pi x_0 k}$ impulsive $\Rightarrow$ periodic

④ $Ш(x) \Rightarrow Ш(k)$

· self transform

impulsive & Periodic $\Rightarrow$ impulsive & Periodic

Properties of Fourier Transform

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scaling $\qquad f(x) \Rightarrow F(k)$

(1) $f(ax) \Rightarrow \frac{1}{|a|} F\left(\frac{k}{a}\right)$ shrink $\Leftrightarrow$ expand, expand $\Leftrightarrow$ shrink

shift

(2) $f(x-x_0) \Rightarrow e^{-i2\pi x_0 k} F(k)$ shift $\Leftrightarrow$ modulation

modulation

③ $e^{-i2\pi k_0 x} f(x) \Rightarrow F(k-k_0)$

symmetry

④ real + even $\Rightarrow$ real + even

real + odd $\Rightarrow$ imag + odd

$f(x)$ real $\Rightarrow F(-k) = F^*(k)$?

what is it good for?

convolution

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⑤ $f(x) * g(x) \Rightarrow F(k) \cdot G(k)$

$f(x) \cdot g(x) \Rightarrow F(k) * G(k)$

why useful?

$$\mathcal{F}\{\underbrace{f(x) * g(x)}_{\text{hard}}\} = \underbrace{F(k)G(k)}_{\text{easy}}$$

why Powerful:

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$$g(x) \longrightarrow \boxed{h(x)} \longrightarrow y(x)$$

often, want specific output so need to design $g(x)$ ⇒ "Deconvolution"

easy with FT

$$Y(k) = G(k)H(k)$$

$$G(k) = \frac{Y(k)}{H(k)}$$

what is the problem?

EXAMPLE:

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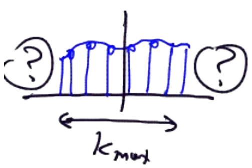
suppose $F(k)$ known over limited range



Probelms: ① support

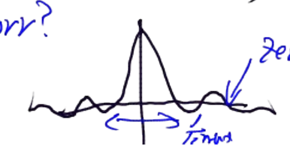
② sampling → later

how do we get back $f(x)$?

① we know $F(k)\Pi\left(\frac{k}{k_{max}}\right)$

$\xrightarrow{FT^{-1}}$ $f(x) * k_{max}\,\mathrm{sinc}(k_{max}x)$ blurr!

(Q) can you deblurr?



Less blurr $k_{max}\uparrow$