Sampling

IDEA: How frequently do you need to sample?

Looks RIGHT!
Sampling:

**IDEA:** How frequently do you need to sample?

Too slow!
(looks constant)
Sampling

#1 is right, but seems over excessive
#2 is too slow
What is the minimum rate?

In general, rate > 2 \cdot BW

> 2 \text{ sample period}
sampling in k-space

In MRI $F(k)$ is sampled over a limited range.

Problems:

Now $\rightarrow$ ① DISCRETE SAMPLING

② FINITE SUPPORT
sampling in $k$-space

Recall:

$$f(k) \cdot \delta(k-k_0) = f(k_0) \delta(k-k_0)$$

Model sampling using impulse train.

$$\left[ \sum_{k=-\infty}^{\infty} \delta(k-n\Delta k) \right] f(k) = \frac{1}{\Delta k} \sum_{i=-\infty}^{\infty} f(k)$$
Recall:

\[ F\left(\frac{k}{a}\right) \Rightarrow k \cdot f(ax) \]

So,

\[ \frac{1}{\Delta k} \mathcal{L}^{-1}\left(\frac{k}{\Delta k}\right) \Rightarrow \mathcal{L}^{-1}\left(\Delta k x\right) \]

also recall

\[ \hat{F}(k) \cdot \hat{G}(k) \Rightarrow f(x) \ast g(x) \]
So, \( \hat{F}(k) = F(k) \cdot \frac{1}{dF} \mathcal{U} \left( \frac{k}{dF} \right) \)

Then,

\[
f(x) = F^{-1} \left\{ \hat{F}(k) \right\} = f(x) \ast \mathcal{U} (dFx)
\]

Recall:

\[
f(x) \ast \delta(x-x_0) = f(x-x_0)
\]
If $f(x)$ is **space limited**:

$$f(x) = \hat{f}(x) \ast \mathbb{U}(\Delta k \cdot x)$$

$\mathbb{U}$ represents the unit step function.

$\mathbb{U}(\Delta k \cdot x)$ is used to represent the field of view (FOV) in the spatial domain.

The diagram illustrates the concept of spatial limited functions and the effect of the unit step function in the spatial domain.
If \( f(x) \) is **SPACE LIMITED**:

\[
\hat{f}(x) = f(x) \ast \mathcal{U}(\Delta k x)
\]

\( \mathcal{U}(\Delta k x) \) is the sinc function, which represents the spatial frequency response.

**Aliasing**: When \( \frac{1}{\Delta k} < \text{FOV} \), the frequency components of \( f(x) \) are folded back into the visible range, leading to aliasing.

\( \text{FOV} \) is the field of view, indicating the range of spatial frequencies that are considered visible or relevant.
Aliasing in x due to large skew
**Sampling in k-space**

In MRI, $F(k)$ is sampled over a limited range.

Problems:

1. Discrete sampling
2. Finite support
Finite support sampling

\[ F_w(k) = F(k) M\left( \frac{k}{W_k} \right) \]

"windowed"

\[ f_w(x) = f(x) \ast W_k \text{ sinc}(W_k x) \]

Full width half max

(FWHM)

\[ = \frac{1}{W_k} \]

Blurr

Ringing
If $f(x)$ is

\[ f(x) = f(x) \ast W_k \text{sinc}(W_k x) \]
Rank the "strength" of artifact due to k-space truncation

\[ f(x) \quad F(k) \quad f_w(x) \]
Discrete finite sampling

Together:

\[ \hat{f}_W(k) = F(k) \frac{\partial}{\partial k} \mathcal{W}(\frac{k}{W_{le}}) \mathcal{W}(\frac{k}{W_{le}}) \]

\[ \hat{f}_W(x) = f(x) * \mathcal{W}(\Delta kx) * W_k \text{sinc}(W_{le}x) \]

Display

Replication

Blur + ripple

Still continuous...
...But images are discrete pixels... so $\hat{f}_W(x)$ is sampled.

$f_W(x) \leftrightarrow \hat{f}_W(k)$

$f_W[n] \leftrightarrow \text{DFT} \rightarrow F_W[m]$
DFT (centered)

\[ F[m] = \sum_{n=-N/2}^{N/2-1} f[n] e^{-i \frac{2\pi}{N} mn} \]

\[ f[n] = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} F[m] e^{i \frac{2\pi}{N} mn} \]

Fast computation with FFT

MATLAB:

\[
\text{DFT} = \text{fftshift}(\text{fft}(\text{fftshift}(\text{signal})))
\]

\[
\text{IDFT} = \text{fftshift}(\text{ifft}(\text{ifftshift}(\text{signal})))
\]
This is a spatial harmonic

\[ f(x,y) = \cos \left( \frac{2\pi}{3x+y} \right) \]
1D FOURIER TRANSFORMS

\[ F(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i \pi (k_x x + k_y y)} \, dx \, dy \]

Q: what is the 2D-Fourier transform of

\[ f(x, y) = \cos(2\pi(3x+y)) \]

A:

\[ \{(s, s), (s, -s), (-s, s), (-s, -s)\} \]
separable functions

\[ f(x, y) = f(x) f(y) \]

Q. Which is the \( 2D \)-FT?

A. (a) \( F(k_x) \ast F(k_y) \)
   
   (b) \( F(k_x) \cdot F(k_y) \)
Sampling & truncation

\[ \hat{F}_W(k_x, k_y) = F(k_x, k_y) \cdot \frac{1}{\Delta k_x \Delta k_y} \cdot W \left( \frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y} \right) \cdot \mathcal{M} \left( \frac{k_x}{W_x}, \frac{k_y}{W_y} \right) \]

Aliasing

\[ \hat{f}_W(x, y) = f(x, y) \ast \ast W \left( \Delta k_x x, \Delta k_y y \right) \ast \ast W_x W_y \cdot \text{sinc} \left( \frac{x}{W_x}, \frac{y}{W_y} \right) \]

More degrees of freedom for sampling!
Given: \( f(x,y) \)

What is \( \Delta k_x, \Delta k_y \)?

(to avoid aliasing)
2D SAMPLING EXAMPLE

GIVEN \( f(x, y) \):

What is \( \Delta k_x, \Delta k_y \)?

(to avoid aliasing)
f(x, y) = f(r)

\[ \mathcal{F}_{1D} \{ f(r) \} = 2\pi \int_0^\infty r f(r) J_0(2\pi rf) dr \]

Hankel transform of zero-th order

... also circularly symmetric

\[ M(r) \Rightarrow J_0(\pi f_k) \approx \frac{\sin c(f_k)}{2f_k} \]

like a sinc... but different
\frac{J_1(\pi x)}{2x} \quad \frac{\sin(\pi x)}{\pi x}
\[
\frac{J_1(\pi r)}{2r}
\]