Assignment 1 Solutions

1. Read Nishimura Ch. 1 and Ch. 2.

2. Consider an old-fashioned LP gramophone phonograph. These were the preferred media for recording music from about 1910 to the late 1980’s. Wikipedia has an interesting entry on this media: http://en.wikipedia.org/wiki/Vinyl_record. Below you can see a close in shot of 8 grooves on a 78 RPM recording, obtained from an LBNL Report 51983, 26-March-2003 Vitaliy Fadeyev and Carl Haber, (publication on the class website). The transverse undulations are transduced into music.

(a) What is the spatial frequency of a 2 kHz tone at the outside radius (4.75 inches)?

Solution:
\[ v_{outer} = \frac{20}{38.799} \text{ kHz/sec} = 51.5 \text{ in}^{-1} \]

(b) What is the spatial frequency of a 2 kHz tone at the inside radius (1.875 inches)?

Solution:
\[ v_{inner} = \frac{20}{15.315} \text{ kHz/sec} = 130.6 \text{ in}^{-1} \]


(a) Which waveform on the right is represented best with a limited number of coefficients? Why?

Solution:
The triangle waveform is best represented with a limited number of coefficients. This is because the triangle looks the most periodic and like a sign wave. Also, the other shapes such as the...
rectangular and bipolar pulse need high frequency components (i.e. more coefficients) to model the sharp edges. This can also be seen from the magnitude spectrum plots which show a rapid drop off for triangle $1/x^2$ and a lower dropoff for the other shapes ($1/x$ for rectangle)

(b) Increase the number of harmonics to represent a rectangular pulse. Notice the Gibb's ringing at the edges. Does the amplitude change much as you vary the number of coefficients?

**Solution:**

The Gibb's ringing always occurs regardless of the number of coefficients used. The Gibb's ringing phenomenon reflects the difficulty in modeling a discontinuous function with a finite number of harmonics. As the number of coefficients increases, the ringing centers at the discontinuities.

(c) The approximation is the inverse FT of a windowed version of the FT of the rectangular pulse. More coefficients means a broader window in k-space. Explain the Gibb's ringing in terms of the FT of this window.

**Solution:**

As was discussed in lecture, we can thing of the $k$-space representation as:

$$F(k) \cap (k_x/W)$$

where $\cap(k_x/W)$ is a windowing function of width $W$. When we inverse FT back to the space domain we get:

$$\mathcal{F}^{-1}\{F(k) \cap (k_x/W)\} = \hat{f}(x) = f(x) * W \text{sinc}(Wx)$$

where $\hat{f}(x)$ is an approximated version of the "perfect" $f(x)$. As $W \to \infty$ the sinc approaches an impulse and so our reconstruction is almost perfect. Thus the width of the Gibbs ringing is reduced. However, the height of $Wsinc(Wx)$ increases as $W \to \infty$ which is why the amplitude of the ringing persists.
4. Proofs:

(a) Prove the stretch or scaling theorem for 2D Fourier Transforms. That is, show that

\[ \mathcal{F}\{f(x/a, y/b)\} = |ab|F(ak_x, bk_y). \]

Solution:

\[ \mathcal{F}\left[f\left(\frac{x}{a}, \frac{y}{b}\right)\right] = |ab|F(ak_x, bk_y) \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(\frac{x}{a}, \frac{y}{b}\right)e^{(-i2\pi k_x x)}e^{(-i2\pi k_y y)}dxdy \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |ab|f\left(\frac{x}{a}, \frac{y}{b}\right)e^{(-i2\pi ak_x u)}e^{(-i2\pi bk_y v)}d\left(\frac{x}{a}\right)d\left(\frac{y}{b}\right) \]
\[ \text{let } u = \frac{x}{a} \]
\[ \text{let } v = \frac{y}{b} \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |ab|f(u, v)e^{(-i2\pi ak_x u)}e^{(-i2\pi bk_y v)}dudv \]
\[ = |ab|\mathcal{F}\left[f(u, v)\right] \]
\[ = |ab|F(ak_x, bk_y) \]

(b) Derive the shift theorem for 2D FTs.

Solution:

\[ \mathcal{F}\left[f(x - a, y - b)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - a, y - b)e^{(-i2\pi k_x x)}e^{(-i2\pi k_y y)}dxdy \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - a, x - b)e^{(-i2\pi k_x (x-a))}e^{(-i2\pi k_y (y-b))}e^{-i2\pi k_x a}e^{-i2\pi k_y b}d(x-a)d(y-b) \]
\[ \text{let } u = x - a \]
\[ \text{let } v = y - b \]
\[ = e^{-i2\pi k_x a}e^{-i2\pi k_y b}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u, v)e^{(-i2\pi k_x u)}e^{(-i2\pi k_y v)}dudv \]
\[ = e^{-i2\pi k_x a}e^{-i2\pi k_y b}F(k_x, k_y) \]

(c) Derive the convolution theorem for 2D FTs.

Solution:
\[ \mathcal{F} [f(x, y) * g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) * g(x, y) e^{-i2\pi k_x x} e^{-i2\pi k_y y} dxdy \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u, v) g(x - u, y - v) dudv \right] e^{-i2\pi k_x x} e^{-i2\pi k_y y} dxdy \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u, v) \left[ \int_{-\infty}^{\infty} G(k_x, k_y) e^{(-i2\pi k_x u)} e^{(-i2\pi k_y v)} dudv \right] \]
\[ = G(k_x, k_y) \int_{-\infty}^{\infty} f(u, v) e^{(-i2\pi k_x u)} e^{(-i2\pi k_y v)} dudv \]
\[ = F(k_x, k_y) G(k_x, k_y) \]

(d) Derive the derivative theorem for 1D FTs. That is, find a simple expression for the FT of \( \frac{df(x)}{dx} \) in terms of \( F(k) \).

Solution:

\[ \mathcal{F} \left[ \frac{df(x)}{dx} \right] = \mathcal{F} \left[ \frac{d}{dx} f(x) \right] \]
\[ = \mathcal{F} \left[ \frac{d}{dx} \int_{-\infty}^{\infty} F(k) e^{(i2\pi k x)} dk \right] \]
\[ = \mathcal{F} \left[ \int_{-\infty}^{\infty} (i2\pi k) F(k) e^{(i2\pi k x)} dk \right] \]
\[ = (i2\pi k) F(k) \]

5. Bandpass Sampling in \( k \)-space: Suppose you have an image \( f(x) \) in 1D x-space that has "finite support" over a region of length \( L \). Finite support simply means that the function is zero outside the region of length \( L \). Here we want to consider the effect of an origin shift. Suppose the support region is shifted away from the origin by a distance \( x_0 \), as shown in the diagram below.

We are going to sample \( F(k_x) \), the functions' FT in \( k_x \) space. Suppose you are asked to specify the minimum sampling requirements in 1D Fourier space (e.g., \( k \)-space).

Solution:
Remember that since we are sampling in \( k \)-space the replication will be in the signal space, e.g., \( x \)-space.
(a) Use the shah function in k-space from section 2.4 of Nishimura to show that the sampling requirement is independent of $x_0$, and only depends on $L$.

**Solution:**
Recall that the center of our signal is at $x = x_0$. From Nishimura Eq. 2.20 we get,

$$\hat{f}(x) = \frac{1}{\Delta k_x} \sum_{n=-\infty}^{\infty} f \left( x - x_0 - \frac{n}{\Delta k_x} \right).$$

All of the replication will be shifted by $x_0$, but they will still be separated by $\frac{1}{\Delta k_x}$.

(b) Find the minimum sampling rate in k-space.

**Solution:**

The minimum sampling rate is

$$\Delta k_x < \frac{1}{L}$$

(c) Would your interpolation filter require knowledge of $x_0$ to get back $F(k_x)$ (and thus $f(x)$) from its samples?

**Solution:**
Yes, the filter requires knowledge of $x_0$, because to select one of the replicated versions, the filter would have to be centered at the center of one of the replicas. For example, at $x_0$ (See above plot).
6. Fourier Transforms and signals:

(a) Consider the Rect function
\[ \cap(x) = \begin{cases} 1 & \text{if } |x| \leq 0.5 \\ 0 & \text{otherwise} \end{cases} \]
and the triangle function
\[ \Lambda(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \]
what is the Fourier transform of \( \cap(x/a) * \cap(x/a) \)?

**Solution:**
We can use the scaling and the convolution property of the Fourier transform
\[
\mathcal{F}\{\cap(x/a) * \cap(x/a)\} = a \cdot \text{sinc}(ax) \cdot \text{sinc}(ax) = a^2 \cdot \text{sinc}^2(ax)
\]
Recall that \( \cap(x) * \cap(x) = \Lambda(x) \).
Since the area of \( \cap(x/a) \) is \( a \) we get,
\[ \cap(x/a) * \cap(x/a) = a \Lambda(x/a) \]
So,
\[
\mathcal{F}\{a\Lambda(x/a)\} = a^2 \cdot \text{sinc}^2(ax)
\]

(b) Consider the sinc function \( \text{sinc} = \frac{\sin(\pi x)}{\pi x} \). What is the transform of the following shifted scaled sinc function?

![Sinc function graph]

**Solution:**
The sinc is shifted by \( c \), scaled by \( a \) and stretched by \( b \). Therefore we get,
\[ f(x) = a \cdot \text{sinc}(b(x - c)) \]
We can use the stretch and the shift theorem to calculate the Fourier transform
\[
\mathcal{F}\{a \cdot \text{sinc}(b(x - c))\} = a \cdot \frac{1}{b} e^{-i2\pi ck} \cap(k/b)
\]
(c) Find the Fourier transform of the following function:

\[ f(x) = \Lambda(x/2) \ast (\delta(x - 4) + \delta(x + 4)) \]

\[ = 2\Lambda(x/2) \frac{\delta(x - 4) + \delta(x + 4)}{2} \]

The Fourier transform is,

\[ \mathcal{F}\{f(x)\} = 4\text{sinc}^2(2k) \cos(2\pi 4k) \]

Solution:
We have two triangles, one shifted by 4 and the other by -4. We can either use the shift and stretch properties, or the convolution and stretch properties. Here, we will use the latter.

We can express the function as,

\[ f(x) = \Lambda(x/2) \ast (\delta(x - 4) + \delta(x + 4)) \]

\[ = 2\Lambda(x/2) \frac{\delta(x - 4) + \delta(x + 4)}{2} \]

The Fourier transform is,

\[ \mathcal{F}\{f(x)\} = 4\text{sinc}^2(2k) \cos(2\pi 4k) \]

(d) The signal \( x[n] = \cos(2\pi fTn) \) is a sampled sinusoid. What is the sequence \( x[n] \) that has the highest discrete frequency and what is the ratio \( f \cdot T \) that produces it?

Solution:
The highest discrete frequency is the one which the period is the shortest. We get this for \( x[n] = \cos(\pi n) = [\cdots, 1, -1, 1, -1, \cdots] \) and when \( f \cdot T = 1/2 \). Interestingly, for this ratio \( T = 1/(2f) \) which also says that the sampling rate should be twice the highest frequency, \textit{i.e.}, Nyquist!
7. The figures below show a signal $f(x)$ and six other signals derived from it.

Suppose $F(k_x)$ is the Fourier transform of $f(x)$. Express the Fourier transform of the other six signals in terms of $F(k_x)$.

**Solution:**

(a) The signal is flipped so, $\tilde{f}(x) = f(-x)$. We then get that $\tilde{F}(k) = F(-k)$. However, if $f(x)$ is real, then $F(-k) = F^*(k)$, so $\tilde{F}(k) = F^*(k)$.

(b) The signal is shifted right by 1, $\tilde{f}(x) = f(x - 1)$. Using the shift property we get $\tilde{F}(k) = e^{-i2\pi k} F(k)$.

(c) $\tilde{f}(x) = f(x) + f(-x)$. We get $\tilde{F}(k) = F(k) + F(-k)$. Again, in the case when $f(x)$ is real then, $\tilde{F}(k) = F(k) + F^*(k) = 2\text{Real}\{F(k)\}$.

(d) $\tilde{f}(x) = f(x/2)$ so, $\tilde{F}(k) = 2F(2k)$.
(e) The signal is flipped and shifted: \( \tilde{f}(x) = f(-(x-2)) \). So,
\[
\tilde{F}(k) = \{e^{-i2\pi 2k}F(k)\}^* = e^{i2\pi 2k}F^*(k)
\]
\[\tilde{F}(k) = e^{i2\pi 2k}F(-k)\] is also a valid answer.

(f) \( \tilde{f}(x) = f(x) + f(x + 2) \). So,
\[
\tilde{F}(k) = F(k) + e^{i2\pi 2k}F(k) = \left(1 + e^{i2\pi 2k}\right)F(k)
\]

8. 2D Fourier transforms:

(a) Find the Fourier transform \( F(k_x, k_y) \) of \( f(x, y) = \text{sinc}(x)\text{sinc}(y) \).

\textbf{Solution:}
This is a separable function, therefore we take the 1D Fourier transform of each part separately.
\[
F(k_x, k_y) = \cap(x) \cap(y)
\]

(b) For \( f(x, y) = \text{sinc}(x)\text{sinc}(y) \). Find the expression for \( g(x, y) = f(x, y) * * f(x, y) \).

\textbf{Solution:}
This is easy to solve in \( k \)-space.
\[
G(k_x, k_y) = F^2(k_x, k_y) = \cap^2(k_x, k_y) = \cap(k_x, k_y)
\]
\[
\Rightarrow g(x, y) = \text{sinc}(x, y) = f(x, y)
\]

(c) Draw the function \( f(x, y) = \cap(x, y) * * \cap(x, y) \)

\textbf{Solution:}
From left to right and top to bottom the function varies linearly. On the diagonal the function varies as a quadratic function. Here’s a contour plot:

9. \textit{Image Scanner}
Consider the following optical scanning system:
The scanner head is equipped with a light sensor that measures the light intensity (0 to 1). The scanner head moves at a velocity \( v \) while scanning a 1D object \( m_c(x) \). The output of the light sensor is \( s(t) = m_c(vt) \). The measure signal is filtered by an ideal low-pass anti-aliasing filter and then sampled at \( T \) to produce \( m[n] = s(nT) \). The antialiasing filter has a cutoff frequency \( \frac{1}{2T} \) and is adjusted to the sampling interval.

(a) Given that the magnitude spectrum of \( m_c \) is

\[
|M_c(kx)|
\]

and that \( T = 1 \) ms, what is the maximum velocity \( v \) cm/s, such that \( M_c(kx) \) can be reconstructed from \( m[n] \)?

**Solution:**
In order to reconstruct \( m_c(x) \) from \( m[n] \) we need to make sure that the velocity is low enough, such that the sampling rate is faster than the Nyquist-rate. We have the relationship: \( s(t) = m_c(vt) \), so \( S(f) = \frac{1}{v} M_c(\frac{f}{v}) \). The temporal frequency of \( s(t) \) is therefore related to the spatial frequency by \( f = vk_x \) [Hz]. To meet the Nyquist rate we would like, the maximum frequency to be less than half the sampling frequency.

\[
f_{\text{max}} < \frac{1}{2T} = \frac{500}{T} \text{[Hz]}
\]

\[
5|1/\text{cm}|v_{\text{max}} < 500\text{[Hz]}
\]

\[
\Rightarrow v < 100\text{[cm/s]}
\]

(b) Draw the magnitude FT (one period) of \( m[n] \) in the interval for \( v = 50 \) cm/s. Does aliasing occur?

**Solutions:**
The velocity is half than the critical speed of 100[cm/s], which means that the highest frequency is also half than the critical frequency. This means that the highest frequency will correspond to \( \pi/2 \). There is no aliasing.
(c) Draw the magnitude FT (one period) of \( m[n] \) for \( v = 200 \text{ cm/s} \). Does aliasing occur?

**Solutions:**
The velocity is twice than the critical velocity. Therefore some of the signal is going to be filtered out by the anti-aliasing filter. Because the anti-aliasing filter matches the sampling rate, there is no aliasing.

(d) We would like to use the system to scan a \( 20 \times 20 \text{ cm} \) page using raster scanning as shown below. The page contains an image, \( m_c(x, y) \) with the following magnitude spectrum \( |M_c(k_x, k_y)| \):

What is the maximum velocity, \( v \) and the maximum distance between lines \( \Delta y \) that is needed such that \( m_c(x, y) \) can be reconstructed from \( m[n_x, n_y] \)? How many lines do you need?

**Solutions:**
This scanning is very similar to MRI. Here we are sampling in the image domain and our spectrum is band limited. In MRI we sample in the frequency domain and our image is space limited.

We need to consider several effects. First, we sample only a finite range, therefore the resulting spectrum is going to suffer from blurring or ringing. Secondly, we need to consider our sampling density. We have to consider two dimensions. In the horizontal, we have exactly the same problem as before, so \( v = 100[\text{cm/s}] \). In the vertical direction, we are sampling spatially, so the rate need to be twice the highest frequency which is: \( \Delta y = 1/10[\text{cm}] \). To cover 20cm we need 200 lines.

(e) We would like to scan faster by increasing both the velocity \( v \) and \( \Delta y \) by a factor of 1.5. Draw the resulting 2D FT of \( m[n_x, n_y] \) for this case. Does aliasing occur?
Solutions:
When increasing the velocity, aliasing does not occur because there is an antialiasing filter which crops the high frequencies. However, the vertical direction is inherently discrete and we can not implement an anti-aliasing filter. Aliasing will occur in that direction as seen in the figure:

A similar phenomena appears in MRI where one sampling direction is continuous and the other is discrete and results in such images: (From http://mriforyou.blogspot.com)