Sampling:
IDEA: How frequently do you need to sample?

Too slow! (Looks constant)

\( \text{Sampling:} \)

#1 is right, but seems over excessive
#2 is too slow
What is the minimum rate?

In general: rate \( > 2 \cdot \text{BW} \)

Looks right.
sampling in k-space

In MRI, $F(k)$ is sampled over a limited range.

Problems:

Now $\rightarrow$ ① DISCRETE SAMPLING

② FINITE SUPPORT

Recall:

\[ f(k) \ast \delta(k - k_0) = f(k_0) \delta(k - k_0) \]

Model sampling using impulse train.

\[
\sum_{n=-\infty}^{\infty} \delta(k - n \Delta k) f(k) = \\
\frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} \delta \left( \frac{k}{\Delta k} - n \right) f(k)
\]

Recall:

\[ F(\frac{k}{\Delta k}) \Rightarrow \text{input} \]

\[ \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} \delta \left( \frac{k}{\Delta k} - n \right) f(k) \rightarrow \text{output} \]

Recall:

$\frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} \delta \left( \frac{k}{\Delta k} - n \right) f(k)$

So,

if $\hat{f}(k) = F(\frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} \delta \left( \frac{k}{\Delta k} - n \right) f(k))$

then

\[ \hat{f}(x) = \mathcal{F}^{-1} \left\{ \sum_{n=-\infty}^{\infty} \delta \left( \frac{k}{\Delta k} - n \right) f(k) \right\} = f(x) * \text{I}(\Delta k x) \]

Recall:

\[ f(x) \ast \delta(x - x_0) = f(x - x_0) \]
If $f(x)$ is **SPACE LIMITED**:

$$f(x)$$

$\text{FOV (Field of View)}$

$$\hat{f}(x) = f(x) \ast \text{ Lyrics } $$

$$\frac{1}{\Delta k} \leq \text{FOV}$$

Aliasing in $x$ due to large $\Delta k$

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If $f(x)$ is **SPACE LIMITED**:

$$f(x)$$

$\text{FOV (Field of View)}$

$$\hat{f}(x) = f(x) \ast \text{ Lyrics } $$

$$\frac{1}{\Delta k} > \text{FOV}$$

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**Sampling in k-space**

In MRI $f(k)$ is sampled over a limited range

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**Problems:**

1. **Discrete Sampling**
2. **Finite Support**

Now $\rightarrow$ 2. **Finite Support**
Finite support sampling

\[ \hat{f}_w(k) = F(k) \tilde{W}(\frac{k}{W_h}) \]

\[ \Rightarrow \hat{f}_w(x) = \hat{f}(x) \ast W_h \text{sinc}(W_h x) \]

*Window*

Full width half max (FWHM)

\[ = \frac{1}{W_h} \]

**QUIZ**

Rank the "strength" of artifact due to k-space truncation

\[ f(x) \quad F(k) \quad \hat{f}_w(x) \]

1. Blurr
2. Ringing
3. Ripple
4. Display

**Discrete finite sampling**

Together:

\[ \hat{f}_w(k) = F(k) \tilde{W}(\frac{k}{W_h}) \tilde{W}(\frac{k}{W_c}) \]

\[ \Rightarrow \hat{f}_w(x) = \hat{f}(x) \ast H(\Delta x) \ast W_h \text{sinc}(W_h x) \]
13. **DFT (centered)**

\[ F[m] = \frac{1}{N} \sum_{n=-N}^{N-1} f[n] e^{-j\frac{2\pi mn}{N}} \]

\[ f[n] = \sum_{m=-N}^{N-1} F[m] e^{j\frac{2\pi mn}{N}} \]

**Fast computation with FFT**

MATLAB:

DFT = fftshift(fft(fftshift(f)))

IDFT = fftshift(ifft(ifftshift(F)))

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14. **Spatial Harmonics & DFT**

This is a spatial harmonic \((-1,1)\)

\[ f(x,y) = \cos(2\pi(3x+y)) \]

15. **2D Fourier Transforms**

\[ F(k_x,k_y) = \iint f(x,y) e^{-j2\pi(k_xx+k_yy)} \, dx \, dy \]

What is the 2D-FT of \[ f(x,y) = \cos(2\pi(3x+y)) \]

\[ f(x,y) = \cos(2\pi(3x+y)) \]

**Notes:**

- **Period:**
  - Sampling period
  - Fourier period

- **Images:**
  - Discrete pixels
  - Sampling

- **Integer:**
  - \( f[n] \leftrightarrow F[m] \)
19. Which is the 2D-FT?

(a) $F(k_x) \ast F(k_y)$

(b) $F(k_x) \cdot F(k_y)$

18. Sampling & convolution

\[ F_w(k_x, k_y) = F(k_x, k_y) \ast \frac{2\pi}{\Delta k_x \Delta k_y} \mathcal{W}(k_x, k_y) \]

\[ F_w(x, y) = f(x, y) \ast \mathcal{W}(\Delta x, \Delta y) \]

More degrees of freedom for sampling!

Blur

20. 2D Sampling Example

Given $f(x, y)$:

What is $\Delta k_x$, $\Delta k_y$?

(to avoid aliasing)
CIRCULARLY SYMMETRIC

\[ f(x, y) = f(r) \]

\[ \mathcal{H}_0 \{ f(r) \} = \frac{1}{2\pi} \int_0^{2\pi} f(r) J_0(2\pi r\sin(\theta)) \, d\theta \]

Hankel transform of zeroth order
... also circularly symmetric

\[ M(r) \rightarrow \frac{J_0(2\pi r f_0)}{2\pi f_0} \triangleq \text{sinc}(f_0) \]

like a sinc... but different