Assignment 1

Due Friday Jan 31st, 2014, Self Grading Due Monday Feb 3rd, 2014

1. Read Nishimura Ch. 1 and Ch. 2.

2. Consider an old-fashioned LP gramophone phonograph. These were the preferred media for recording music from about 1910 to the late 1980’s. Wikipedia has an interesting entry on this media: http://en.wikipedia.org/wiki/Vinyl_record. Below you can see a close in shot of 8 grooves on a 78 RPM recording, obtained from an LBNL Report 51983, 26-March-2003 Vitaliy Fadeyev and Carl Haber, (publication on the class website). The transverse undulations are transduced into music.

(a) What is the spatial frequency of a 2 kHz tone at the outside radius (4.75 inches)?
(b) What is the spatial frequency of a 2 kHz tone at the inside radius (1.875 inches)?

Figure 1: Micro-photograph of grooves on a 78 r.p.m recording. Illumination is coaxial. Image size is approximately 700 × 540 microns.


(a) Which waveform on the right is represented best with a limited number of coefficients? Why?
(b) Increase the number of harmonics to represent a rectangular pulse. Notice the Gibb’s ringing at the edges. Does the amplitude change much as you vary the number of coefficients?
(c) The approximation is the inverse FT of a windowed version of the FT of the rectangular pulse. More coefficients means a broader window in k-space. Explain the Gibb’s ringing in terms of the FT of this window.
4. Proofs:

(a) Prove the stretch or scaling theorem for 2D Fourier Transforms. That is, show that
\[ \mathcal{F}\{f(x/a, y/b)\} = |ab| F(ak_x, bk_y). \]

(b) Derive the shift theorem for 2D FTs.
(c) Derive the convolution theorem for 2D FTs.
(d) Derive the derivative theorem for 1D FTs. That is, find a simple expression for the FT of \( \frac{df(x)}{dx} \) in terms of \( F(k) \).

5. Bandpass Sampling in \( k \)-space: Suppose you have an image \( f(x) \) in 1D \( x \)-space that has "finite support" over a region of length \( L \). Finite support simply means that the function is zero outside the region of length \( L \). Here we want to consider the effect of an origin shift. Suppose the support region is shifted away from the origin by a distance \( x_0 \), as shown in the diagram below.

![Diagram of bandpass sampling](image)

We are going to sample \( F(k_x) \), the functions' FT in \( k_x \) space. Suppose you are asked to specify the minimum sampling requirements in 1D Fourier space (e.g., \( k \)-space).

(a) Use the shah function in \( k \)-space from section 2.4 of Nishimura to show that the sampling requirement is independent of \( x_0 \), and only depends on \( L \).

(b) Find the minimum sampling rate in \( k \)-space.

(c) Would your interpolation filter require knowledge of \( x_0 \) to get back \( F(k_x) \) (and thus \( f(x) \)) from its samples?
6. Fourier Transforms and signals:

(a) Consider the Rect function

\[ \n(x) = \begin{cases} 
1 & |x| \leq 0.5 \\
0 & \text{otherwise} 
\end{cases} \]

and the triangle function

\[ \Lambda(x) = \begin{cases} 
1 - |x| & |x| \leq 1 \\
0 & \text{otherwise} 
\end{cases} \]

what is the Fourier transform of \( \n(x/a) \ast \n(x/a) \)?

(b) Consider the sinc function \( \text{sinc} = \frac{\sin(\pi x)}{\pi x} \). What is the transform of the following shifted scaled sinc function?

(c) Find the Fourier transform of the following function:

(d) The signal \( x[n] = \sin(2\pi fTn) \) is a sampled sinusoid. What is the sequence \( x[n] \) that has the highest discrete frequency and what is the ratio \( f \cdot T \) that produces it?
7. The figures below show a signal \( f(x) \) and six other signals derived from it.

Suppose \( F(k_x) \) is the Fourier transform of \( f(x) \). Express the Fourier transform of the other six signals in terms of \( F(k_x) \).

8. 2D Fourier transforms:

(a) Find the Fourier transform \( F(k_x, k_y) \) of \( f(x, y) = \text{sinc}(x)\text{sinc}(y) \).

(b) For \( f(x, y) = \text{sinc}(x)\text{sinc}(y) \). Find the expression for \( g(x, y) = f(x, y) \ast \ast f(x, y) \).

(c) Draw the function \( f(x, y) = \Box(x, y) \ast \ast \Box(x, y) \)
9. **Image Scanner**

Consider the following optical scanning system:

The scanner head is equipped with a light sensor that measures the light intensity (0 to 1). The scanner head moves at a velocity \( v \) while scanning a 1D object \( m_c(x) \). The output of the light sensor is

\[
s(t) = m_c(vt) \]

The measure signal is filtered by an ideal low-pass anti-aliasing filter and then sampled at \( T \) to produce

\[
m[n] = s(nT) \]

(a) Given that the magnitude spectrum of \( m_c \) is

\[
|M_c(k_x)| \text{[1/cm]} \]

and that \( T = 1 \text{ ms} \), what is the maximum velocity \( v \) cm/s, such that \( M_c(k_x) \) can be reconstructed from \( m[n] \)?

(b) Draw the magnitude FT (one period) of \( m[n] \) in the interval for \( v = 50 \text{ cm/s} \). Does aliasing occur?

(c) Draw the magnitude FT (one period) of \( m[n] \) for \( v = 200 \text{ cm/s} \). Does aliasing occur?

(d) We would like to use the system to scan a 20 \( \times \) 20 cm page using raster scanning as shown below. The page contains an image, \( m_c(x, y) \) with the following magnitude spectrum \( |M_c(k_x, k_y)| \):

What is the maximum velocity \( v \) and the maximum distance between lines \( \Delta y \) that is needed such that \( m_c(x, y) \) can be reconstructed from \( m[n_x, n_y] \)? How many lines do you need?

(e) We would like to scan faster by increasing both the velocity \( v \) and \( \Delta y \) by a factor of 1.5. Draw the resulting 2D FT of \( m[n_x, n_y] \) for this case. Does aliasing occur?