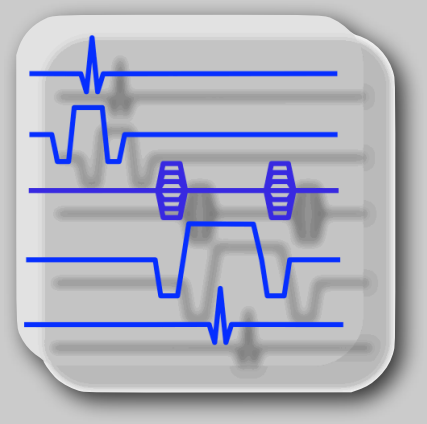


Principles of MRI

EE225E / BIO265

Chapter 05

M. Lustig, EECS UC Berkeley

MRI Haiku

Recon runs all night
k-space, image space now done
oh oh, artifacts!

D. Nishimura 2012

The Bloch Equation

- Combine Precession and relaxation

$$\frac{d\vec{M}}{dt} = \underbrace{-\gamma \vec{B} \times \vec{M}}_{\text{precession}} - \underbrace{\frac{M_x \hat{i} + M_y \hat{j}}{T_2}}_{\text{transverse decay}} - \underbrace{\frac{M_z - M_0}{T_1} \hat{k}}_{\text{long. recovery}}$$

- Phenomenological: Fits observations
  - –Describes most of MRI
  - –Sometimes Fails.... J-coupling, Magn, transfer

Mathematical Description of MRI

- Plan:
  1) Derive Math for each element
  2) Put together : e.g., the BLOCH equation
  3) Solve the Bloch eqn. for special cases
     a) Excitation CH. 6 (later)
     b) Reception CH. 5 (first)
        i) Derive k-space (AGAIN!!!)
        ii) Pulse sequence
        iii)Sampling

The Bloch Equation

- In the rotating frame: $\vec{r} = [x,\ y,\ z]^T$

$$\vec{B} = \gamma \vec{G} \cdot \vec{r}\hat{k} + B_{1x}\hat{i} + B_{1y}\hat{j}$$

$$\frac{d\vec{M}}{dt} = \underbrace{-\gamma \vec{B} \times \vec{M}}_{\text{precession}} - \underbrace{\frac{M_x\hat{i} + M_y\hat{j}}{T_2}}_{\text{transverse decay}} - \underbrace{\frac{M_z - M_0}{T_1}\hat{k}}_{\text{long. recovery}}$$

The Bloch Equation

- In Matrix Form :

$$
\begin{bmatrix} \dot{M}_x \\ \dot{M}_y \\ \dot{M}_z \end{bmatrix} = \begin{bmatrix} 0 & \gamma \vec{G} \cdot \vec{r} & -\gamma B_{1y} \\ -\gamma \vec{G} \cdot \vec{r} & 0 & \gamma B_{1x} \\ \gamma B_{1y} & -\gamma B_{1x} & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \Big\} \text{Precession}
$$

$$
+ \begin{bmatrix} -\frac{1}{T_2} & 0 & 0 \\ 0 & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \Big\} \text{relaxation}
$$

$$
+ \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix} M_0 \Big\} \text{recovery}
$$

Bloch Equation

- Combined (rotating frame)

$$\begin{bmatrix} \dot{M}_x \\ \dot{M}_y \\ \dot{M}_z \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{T_2} & \gamma\vec{G}\cdot\vec{r} & -\gamma B_{1y} \\ -\gamma\vec{G}\cdot\vec{r} & -\frac{1}{T_2} & \gamma B_{1x} \\ \gamma B_{1y} & -\gamma B_{1x} & -\frac{1}{T_1} \end{bmatrix}}_{\text{Rotation + relaxation}} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix} M_0}_{\text{Recovery}}$$

- T1 is <u>the</u> source of all signals!
- Magnetization distribution unknown
- Can probe by changing B1 and G

Solving the Bloch Equation

Two Special Cases:

- Reception $B_{1x} = 0, \; B_{1y} = 0$
  - –Data acquisition
  - –Spatial encoding
  - –Explicit solutions! (Today)
- Excitation $B_{1x} \neq 0, \; B_{1y} \neq 0$
  - –Non linear problems
  - –No general solution
  - –Many solutions for special cases (Ch. 6)

Reception

- Magnetization is a function of $\vec{r}$ and t. Also, assume single, constant T1 and T2.

$$\vec{M}(\vec{r},t) = [M_x(\vec{r},t),\ M_y(\vec{r},t),\ M_z(\vec{r},t)]^T$$

- We would like to resolve this using a time varying gradient

$$\vec{G}(t) \ = \ [G_x(t),\ G_y(t),\ G_z(t)]^T$$

Reception

- With no B1(t) :

$$\begin{bmatrix} \dot{M}_x \\ \dot{M}_y \\ \dot{M}_z \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & \gamma\vec{G}\cdot\vec{r} & 0 \\ -\gamma\vec{G}\cdot\vec{r} & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix} M_0$$

Reception

- With no B1(t) :

$$\begin{bmatrix} \dot{M}_x \\ \dot{M}_y \\ \dot{M}_z \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & \gamma\vec{G}(t)\cdot\vec{r} & 0 \\ -\gamma\vec{G}(t)\cdot\vec{r} & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix} M_0$$

- Decouples into two independent eqns.

① $$\begin{pmatrix} \dot{M}_x \\ \dot{M}_y \end{pmatrix} = \begin{pmatrix} -\frac{1}{T_2} & \gamma\vec{G}(t)\cdot\vec{r} \\ -\gamma\vec{G}(t)\cdot\vec{r} & -\frac{1}{T_2} \end{pmatrix} \begin{pmatrix} M_x \\ M_y \end{pmatrix}$$

② $$\dot{M}_z = -\frac{1}{T_1} M_z + \frac{1}{T_1} M_0$$

Solution for Reception

- Mz: (2) is already solved!

$$M_z(\vec{r},t) = M_0(\vec{r}) + \left[M_z(\vec{r},0) - M_0(\vec{r})\right]e^{-\frac{t}{T_1}}$$

$$= \underbrace{M_z(\vec{r},0)e^{-\frac{t}{T_1}}}_{\text{Initial Magn. DECAYS}} + \underbrace{\left(1-e^{-\frac{t}{T_1}}\right)M_0(\vec{r})}_{\text{EQUILIBRIUM MAGN. RESTORED}}$$

Solution for Reception

- Mxy: (1)

$$\begin{pmatrix} \dot{M}_x \\ \dot{M}_y \end{pmatrix} = \begin{pmatrix} -\frac{1}{T_2} & \gamma\vec{G}(t)\cdot\vec{r} \\ -\gamma\vec{G}(t)\cdot\vec{r} & -\frac{1}{T_2} \end{pmatrix} \begin{pmatrix} M_x \\ M_y \end{pmatrix} \begin{matrix} (a) \\ (b) \end{matrix}$$

Take eqn (a) + i eqn (b)

$$\dot{M}_x + i\dot{M}_y = -\frac{1}{T_2}M_x + \gamma\vec{G}(t)\cdot\vec{r}M_y - i\gamma\vec{G}(t)\cdot\vec{r}M_x - \frac{i}{T_2}M_y =$$

$$= -\frac{1}{T_2}(M_x + iM_y) - i\gamma\vec{G}(t)\cdot\vec{r}\, iM_y - i\gamma\vec{G}(t)\cdot\vec{r}M_x$$

$\boxed{+ \text{is } -i^2}$

Solution for Reception

$$\dot{M}_{xy} = \left(-\frac{1}{T_2} - i\gamma\vec{G}(t)\cdot\vec{r}\right) M_{xy}$$

$$\Rightarrow 0 = \dot{M}_{xy} + \left(\frac{1}{T_2} + i\gamma\vec{G}(t)\cdot\vec{r}\right) M_{xy}$$

- Can be solved using an integrating factor

Differential Equations 101

- Given an ODE

$$y' + p(x)y = Q(x)$$

- Multiply with integrating factor M(x)

- So, $$M(x)y' + M(x)p(x)y = M(x)Q(x)$$

$$(M(x)y)' = M(x)Q(x)$$

- And,

$$y(x)M(x) = \int Q(x)M(x) \; + \; C$$

Solution for Reception

$$\dot{M}_{xy} = \left(-\frac{1}{T_2} - i\gamma\vec{G}(t)\cdot\vec{r}\right) M_{xy}$$

$$\Rightarrow 0 = \dot{M}_{xy} + \left(\frac{1}{T_2} + i\gamma\vec{G}(t)\cdot\vec{r}\right) M_{xy}$$

- Can be solved using an integrating factor

$$e^{\frac{t}{T_2} + i\gamma\vec{r}\cdot\int_0^t \vec{G}(\tau)d\tau}$$

Solution for Reception

$$\dot{M}_{xy} + \left(\frac{1}{T_2} + i\gamma\vec{G}(t)\cdot\vec{r}\right) M_{xy} = 0$$

$$\dot{M}_{xy} e^{\frac{t}{T_2} + i\gamma\vec{r}\cdot\int_0^T \vec{G}(\tau)d\tau} + \left(\frac{1}{T_2} + i\gamma\vec{G}(t)\cdot\vec{r}\right) M_{xy} e^{\frac{t}{T_2} + i\gamma\vec{r}\cdot\int_0^T \vec{G}(\tau)d\tau} = 0$$

$$\left(M_{xy} e^{\frac{t}{T_2} + i\gamma\vec{r}\cdot\int_0^t \vec{G}(\tau)d\tau}\right)' = 0$$

- Integrate:

$$M_{xy}(\vec{r},t) e^{\frac{t}{T_2} + i\gamma\vec{r}\cdot\int_0^t \vec{G}(\tau)d\tau} = M_{xy}(\vec{r},0)$$

Solution for Reception

$$M_{xy}(\vec{r},t) = \underbrace{M_{xy}(\vec{r},0)}_{\text{object}}\underbrace{e^{-\frac{t}{T_2}}}_{\text{decay}}\underbrace{e^{-i\vec{r}\cdot\int_0^t \gamma\vec{G}(\tau)d\tau}}_{\text{spatial encoding}}$$

- Define a Spatial frequency vector:

$$\vec{k}(t) = [k_x(t), k_y(t), k_z(t)]^T$$

$$\boxed{\vec{k}(t) = \frac{\gamma}{2\pi}\int_0^t \vec{G}(\tau)d\tau}$$

Solution for Reception

- Transverse Magnetization Mxy is then:

$$M_{xy}(\vec{r},t) = M_{xy}(\vec{r},0)e^{-\frac{t}{T_2}}e^{-i2\pi\vec{k}(t)\cdot\vec{r}}$$

Fourier kernel

- Received Signal is proportional to Mxy integrated over volume:

$$s(t) = \int_{\vec{R}} M_{xy}(\vec{r},0)e^{-\frac{t}{T_2}}e^{-i2\pi\vec{k}(t)\cdot\vec{r}}d\vec{r}$$

Also a function of r!

Signal Equation

- Assume that T2 is <u>LARGE</u>

$$s(t) \approx \int_{\vec{R}} M_{xy}(\vec{r},0) e^{-i2\pi \vec{k}(t)\cdot\vec{r}} d\vec{r}$$

- So,

$$s(t) = \tilde{\mathcal{F}}\left\{ M_{xy}(\vec{r},0) \right\}\Big|_{\vec{k}=\vec{k}(t)}$$

samples of FT of $M_{xy}(\vec{r},0)$ along a trajectory

k-Space

$$s(t) = \mathcal{F}\left\{ M_{xy}(\vec{r},0) \right\}\Big|_{\vec{k}=\vec{k}(t)}$$

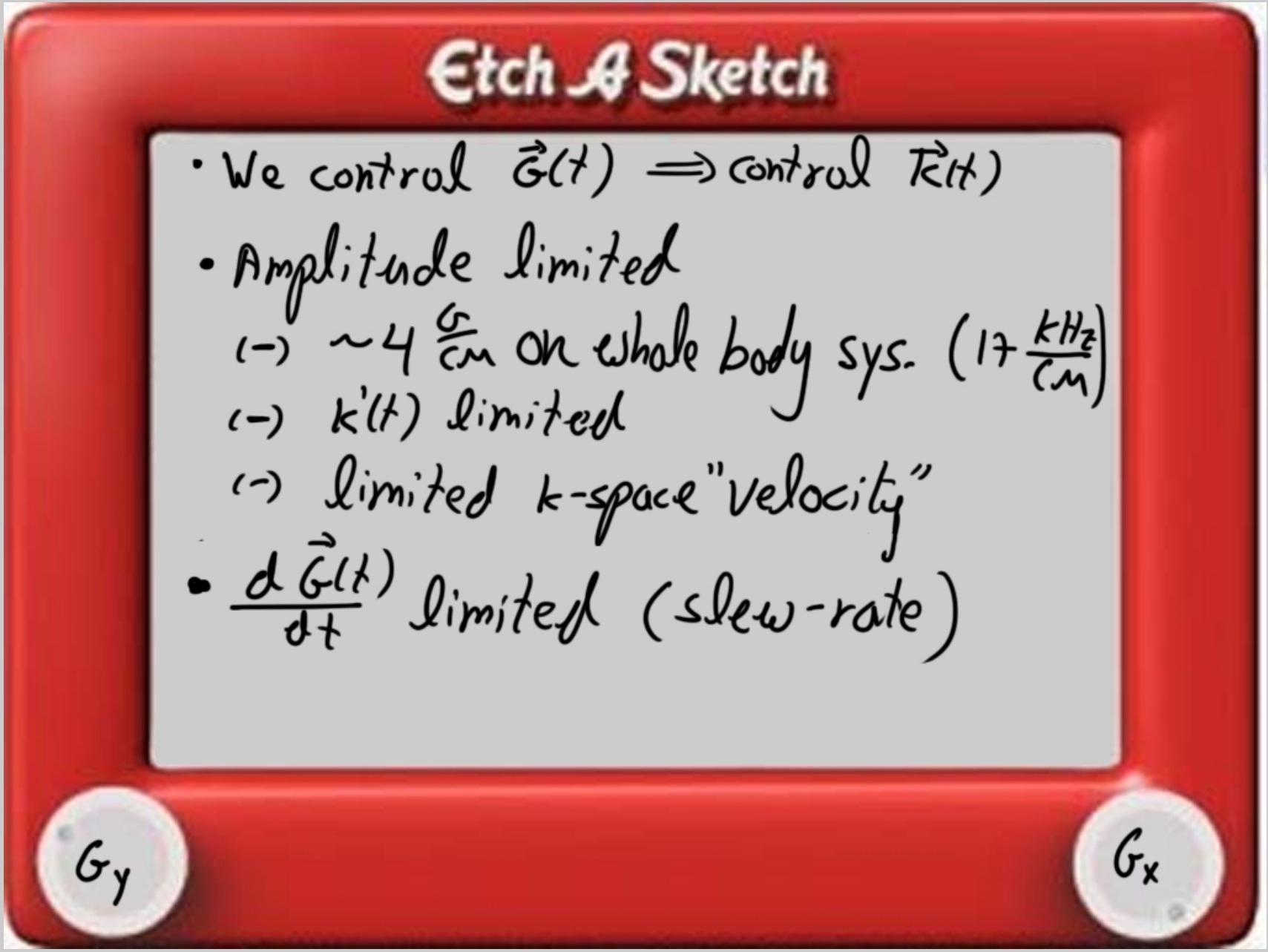
samples of FT of $M_{xy}(\vec{r},0)$ along a trajectory

- $\vec{k}(t)$ is k-space trajectory

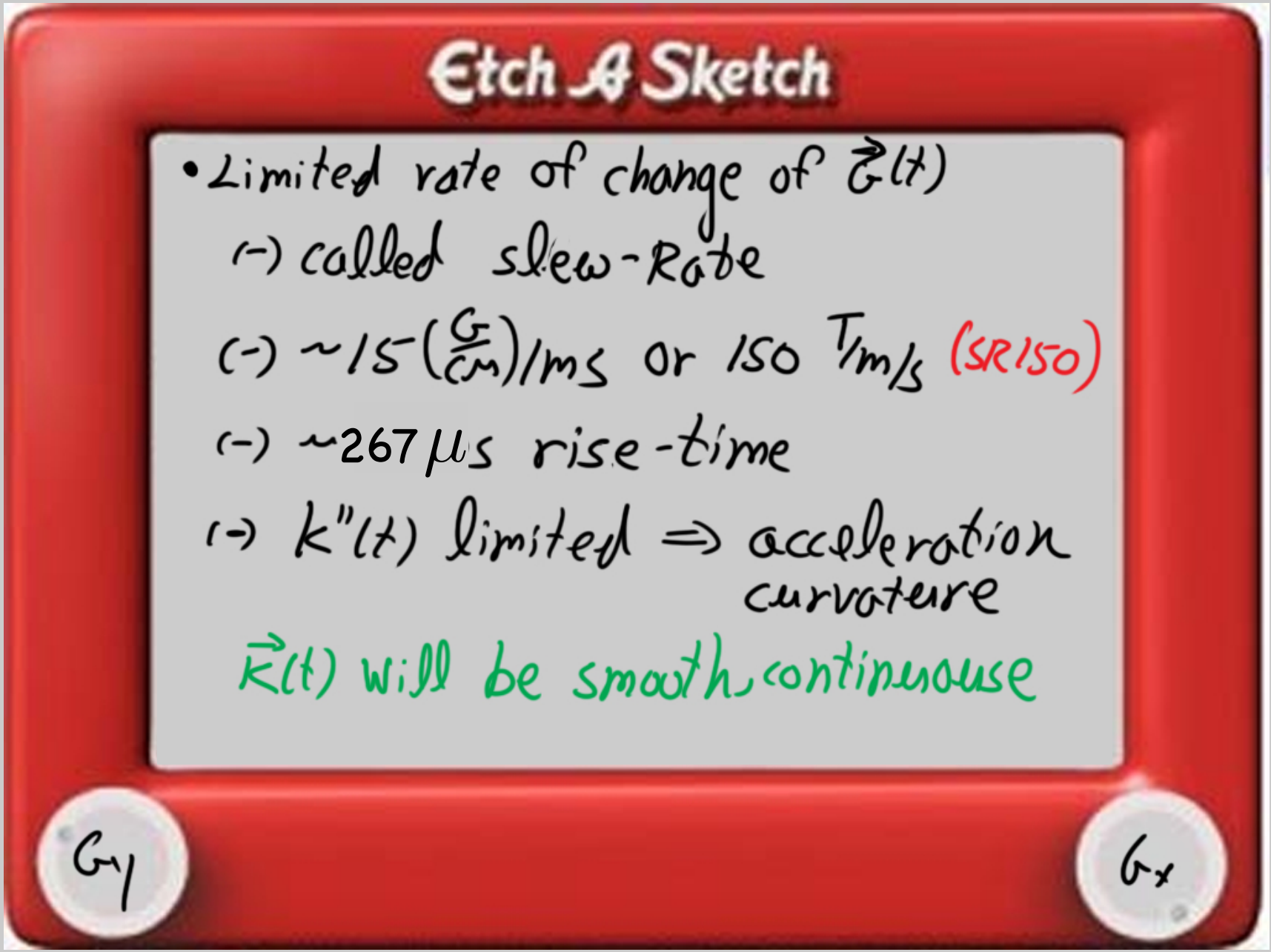
$$\vec{k}(t) = \frac{\gamma}{2\pi}\int_0^t \vec{G}(\tau)\,d\tau$$

⊛ area of gradient waveform

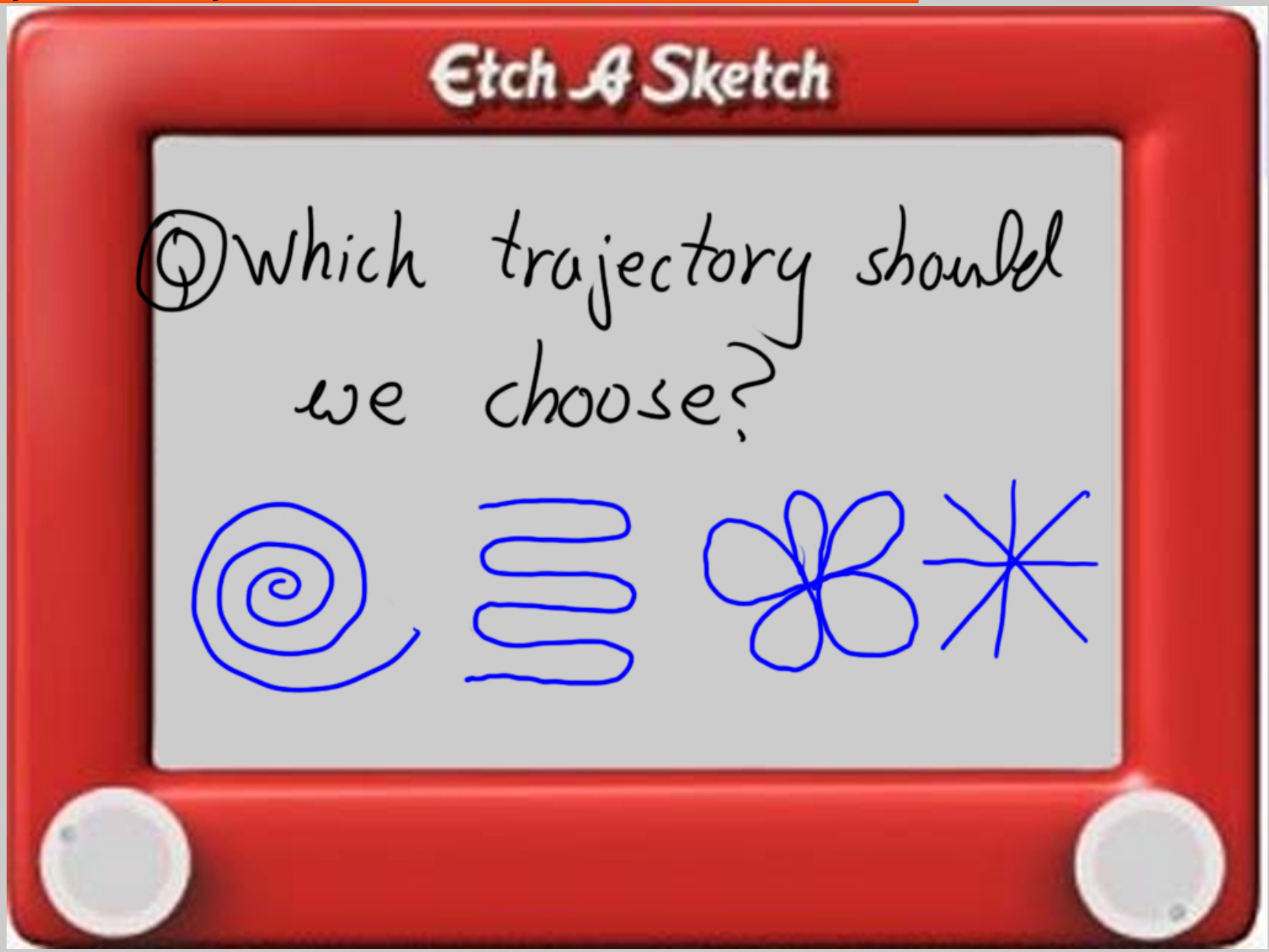
k-Space Trajectories



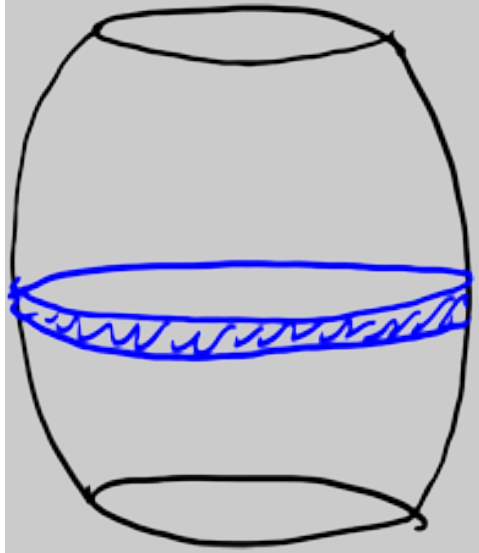
k-Space Trajectories



k-Space Trajectories



Slice Selective 2D Example

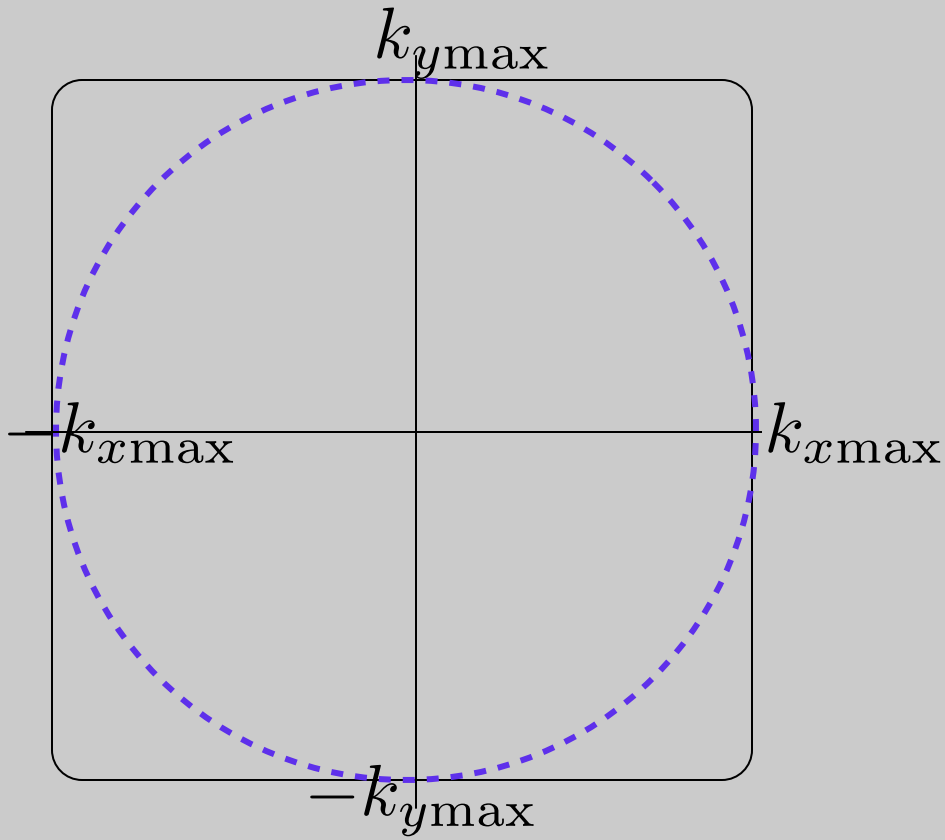


- Slice Select in Z
- Resolve in x,y
- Need $G_x(t)$, $G_y(t)$
- Acquire $k_x(t)$,$k_y(t)$

$$s(t) = \int_x \int_y \int_z M_{xy}(\vec{r},0)\, e^{-i2\pi \vec{k}(t)\cdot\vec{r}}\, d\vec{r}$$

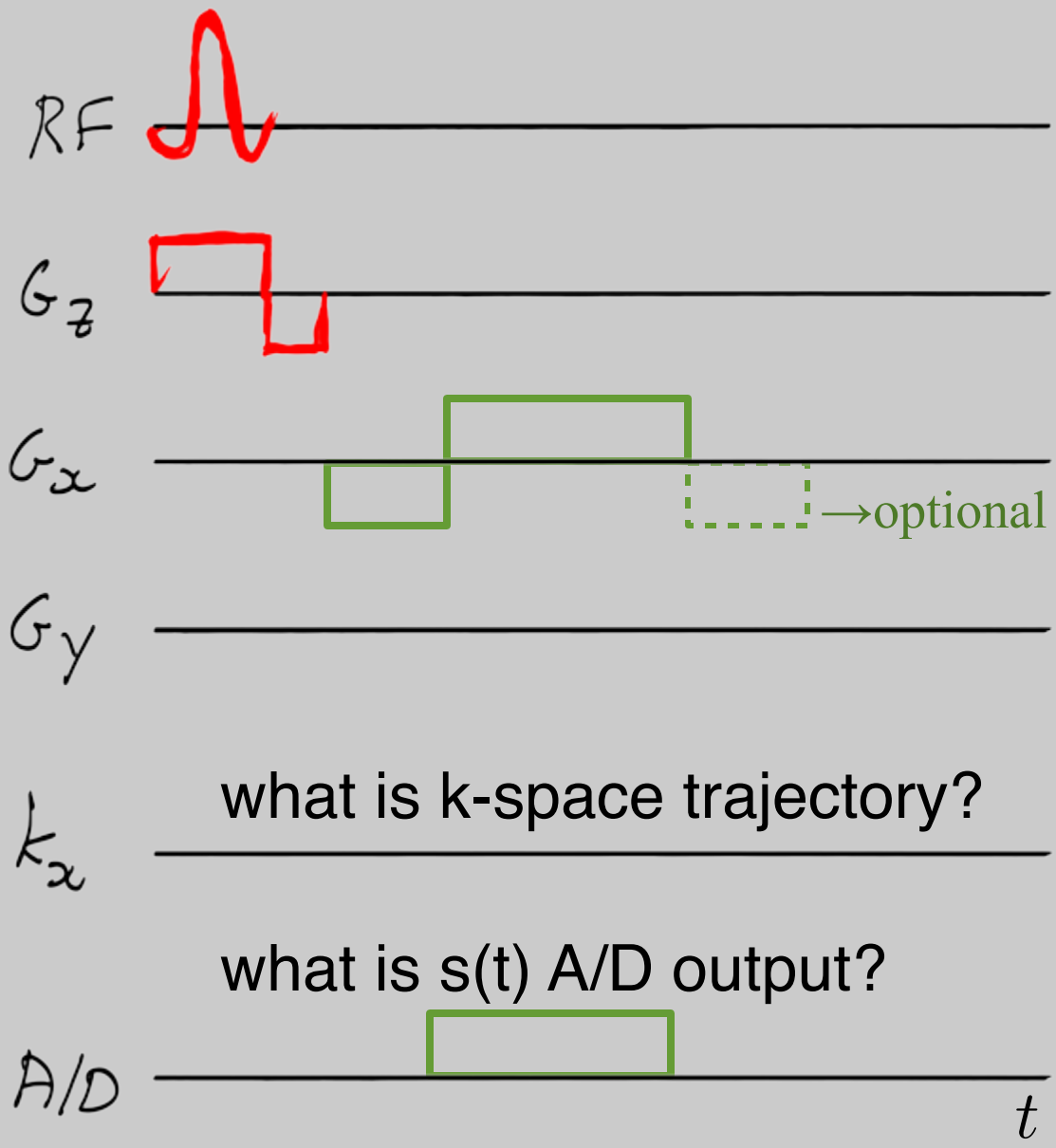
$$k_z(t) = 0 \Longrightarrow \int_x \int_y \left[\int_z M_{xy}(\vec{r},0)\, dz\right] e^{-i2\pi \vec{k}(t)\cdot\vec{r}}\, d\vec{r}$$

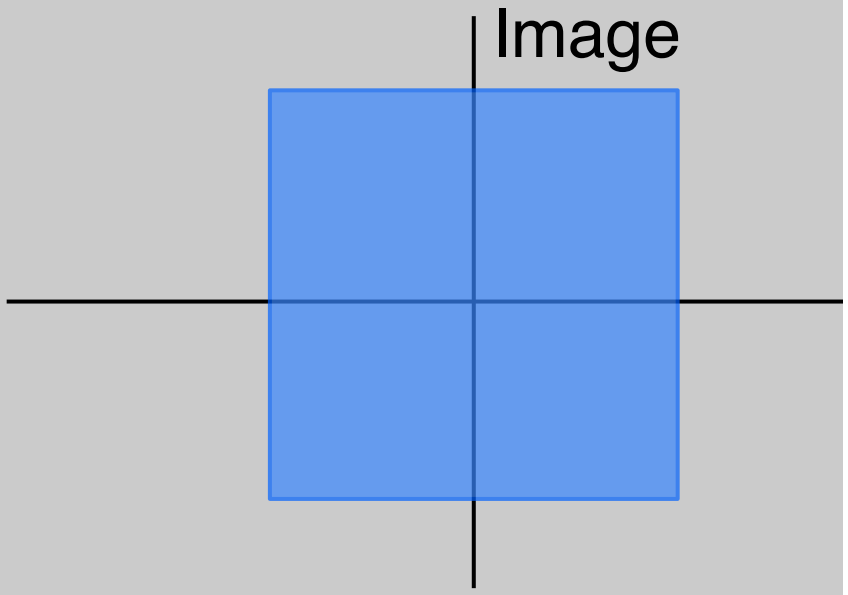
k-Space Trajectory

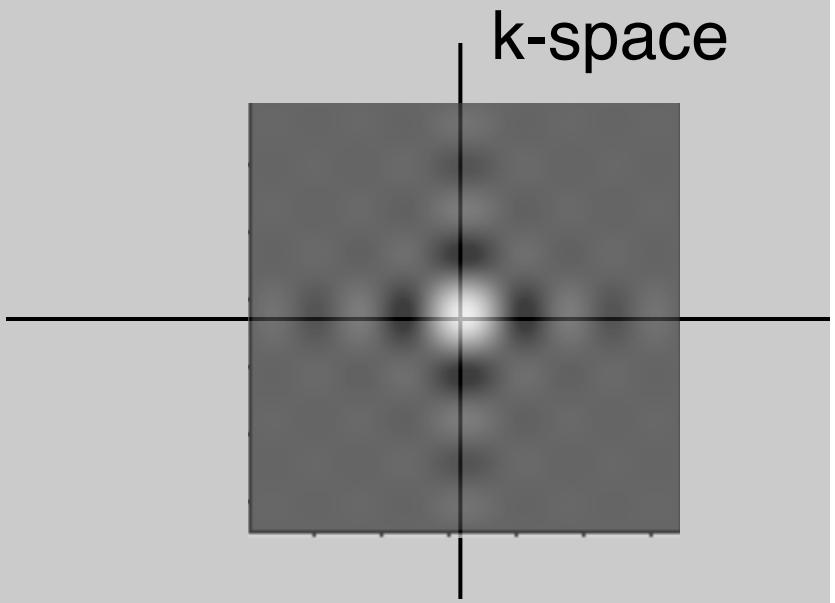


- Extent ⇒ Resolution
- Density ⇒ FOV (next time)
- Often Square coverage
  - Sometimes circle
- What are some options?

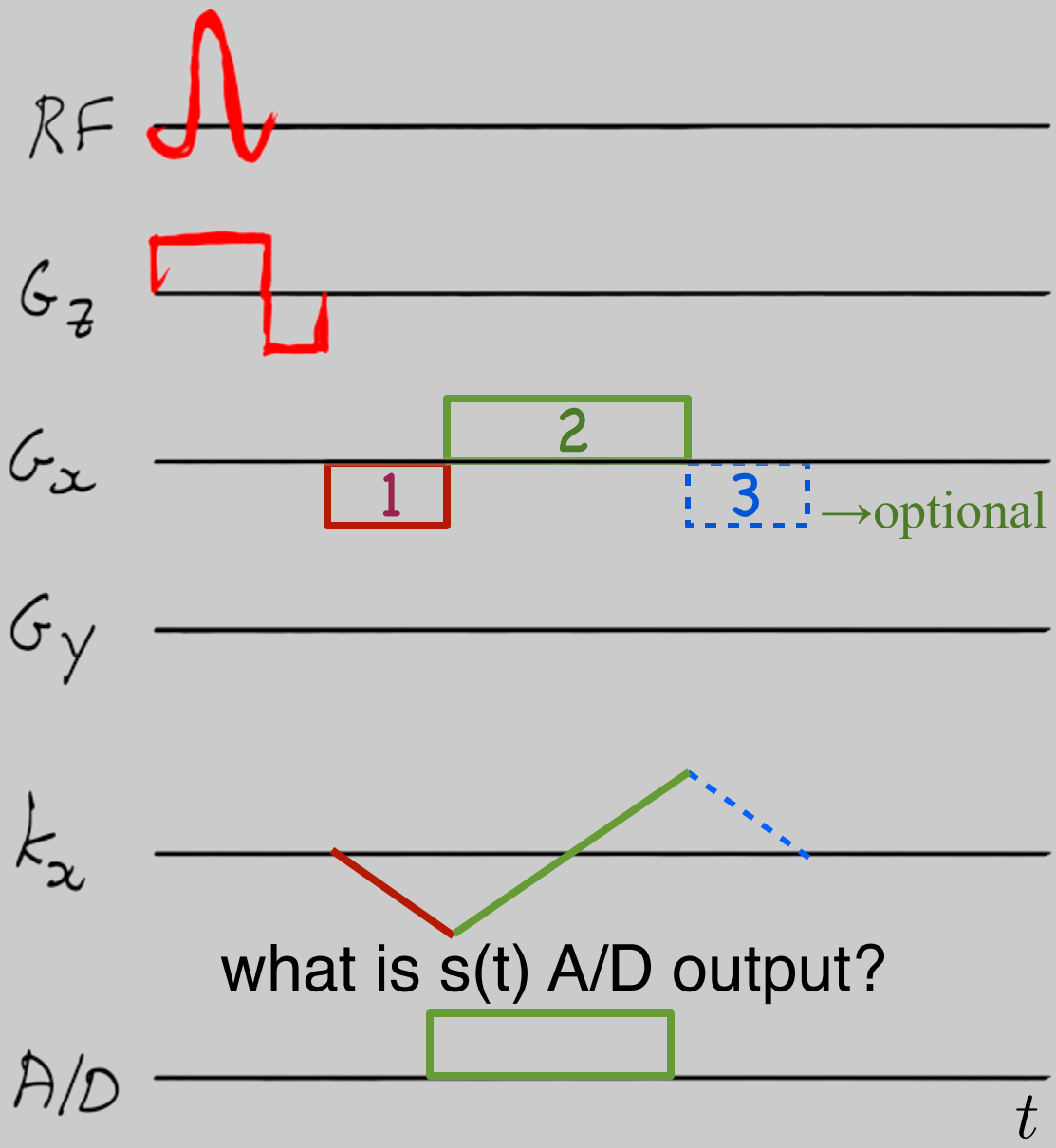
Radial Scanning (Projection Reconstruction)

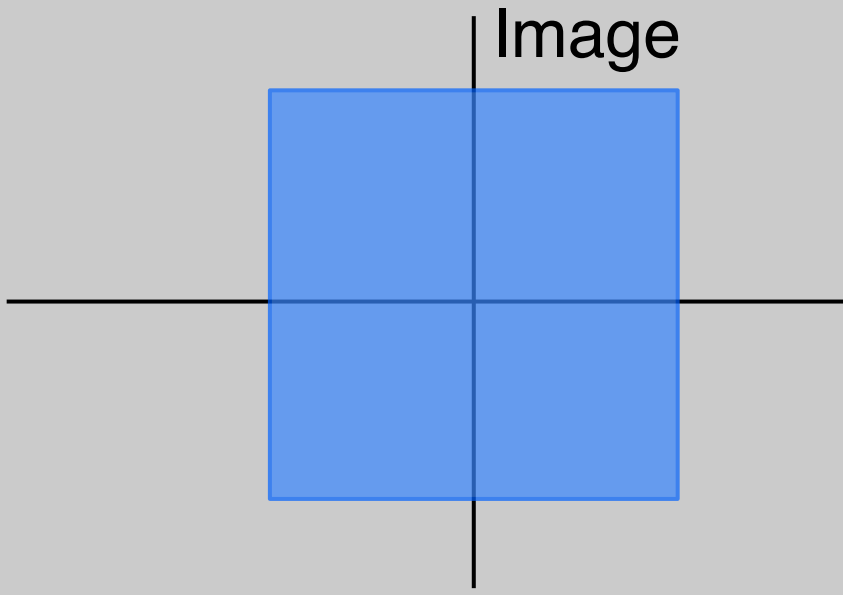


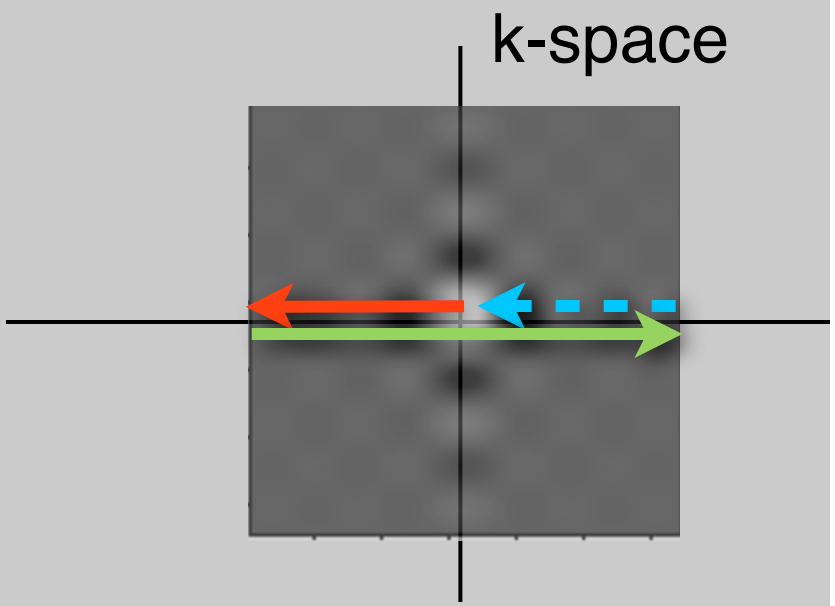




Radial Scanning (Projection Reconstruction)







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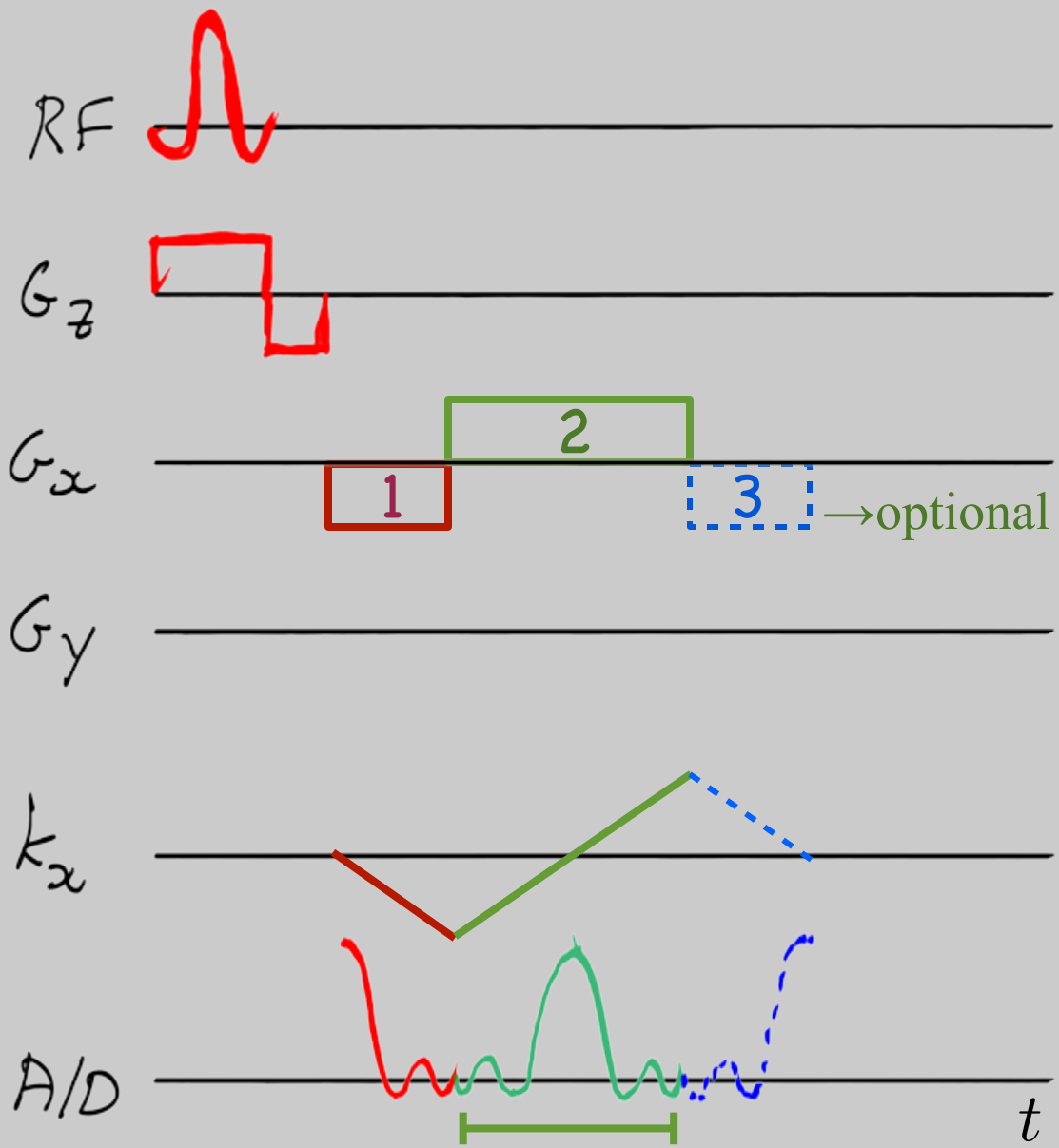
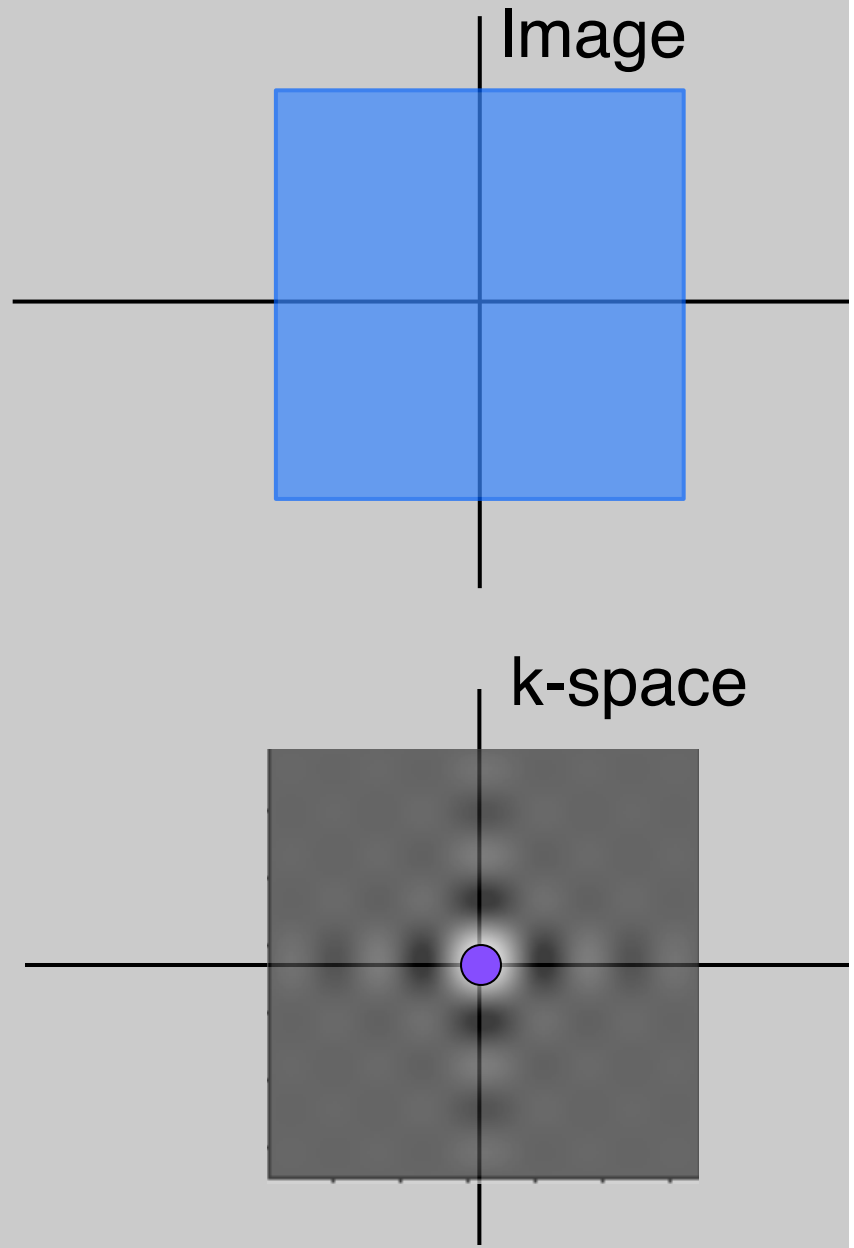
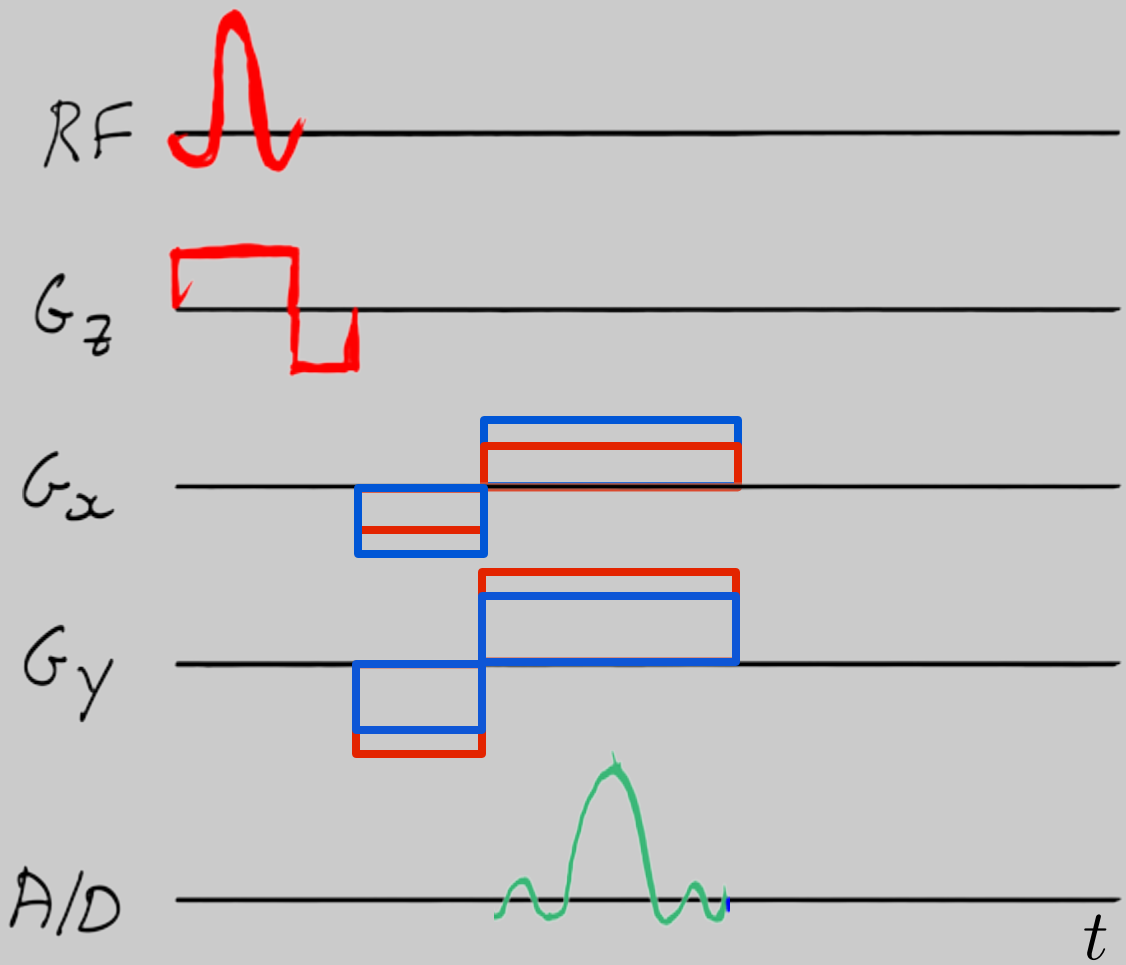
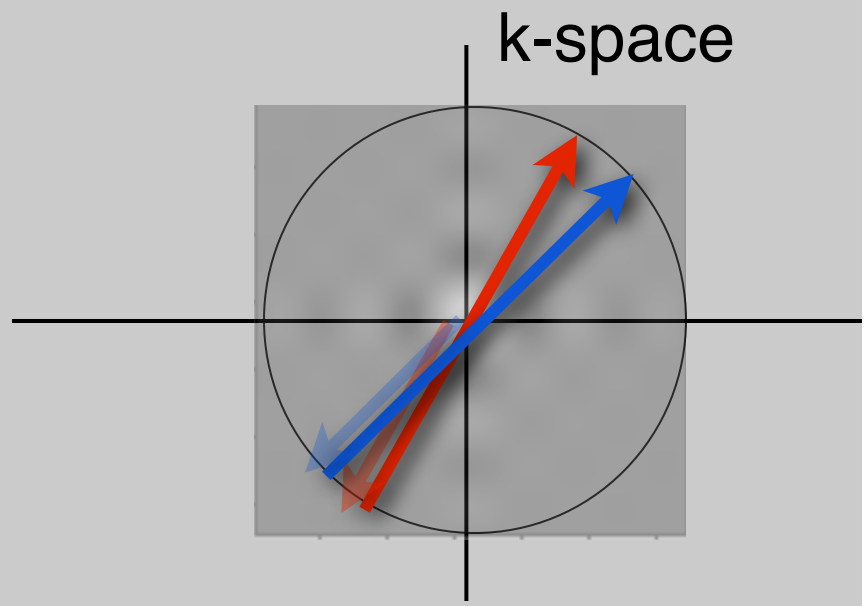




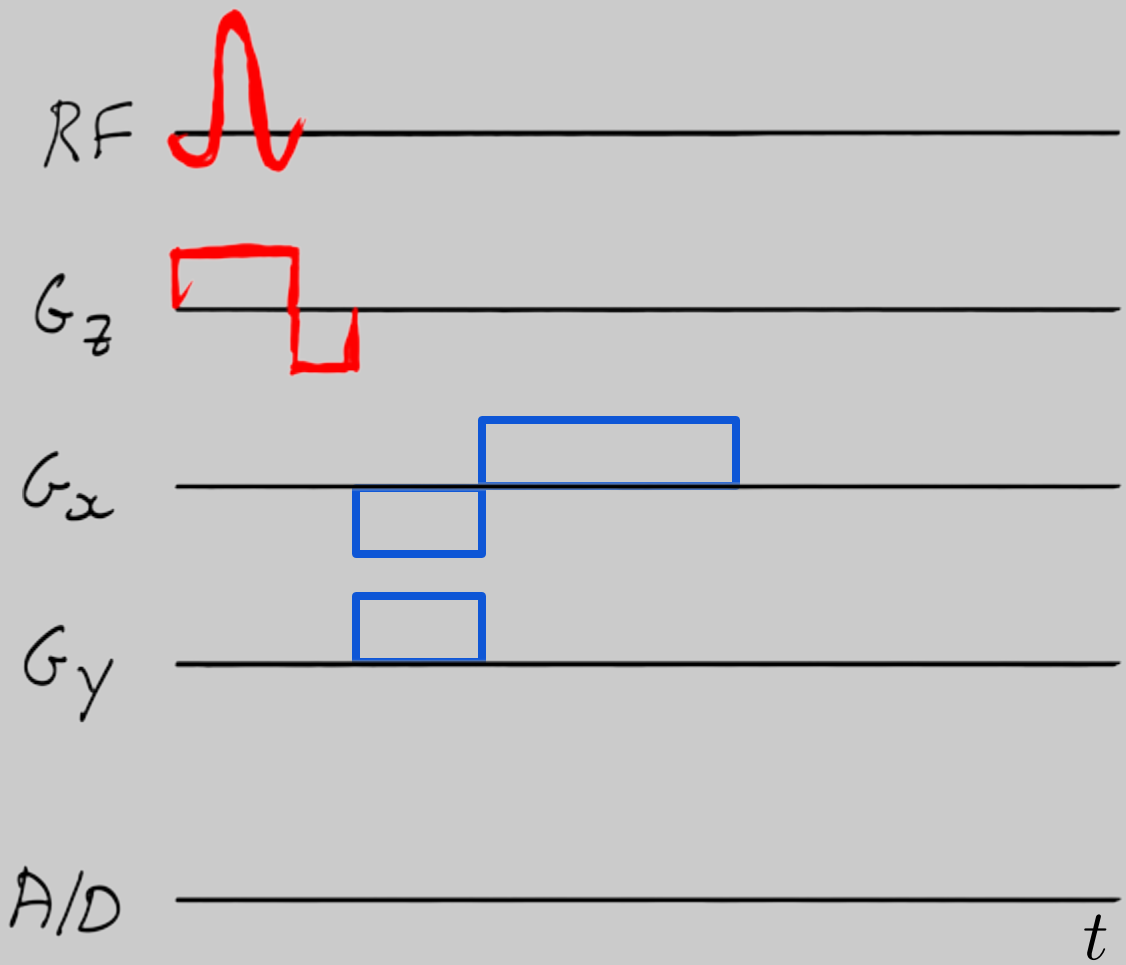
Radial Scanning (Projection Reconstruction)

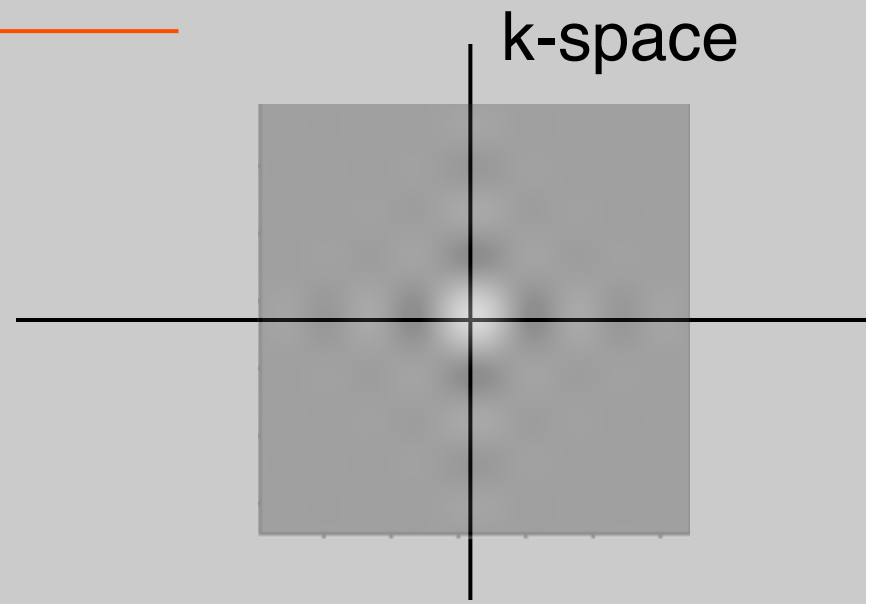


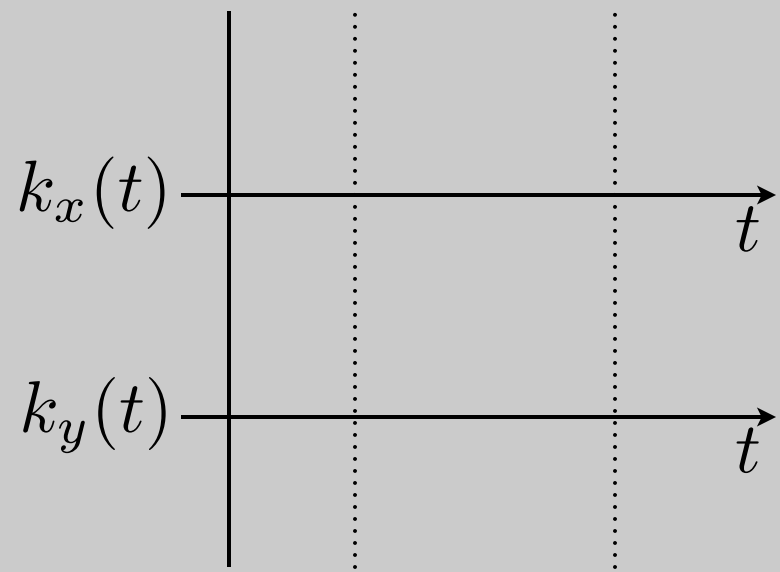


- Repeat for each diameter
- Peak at k(t)=0

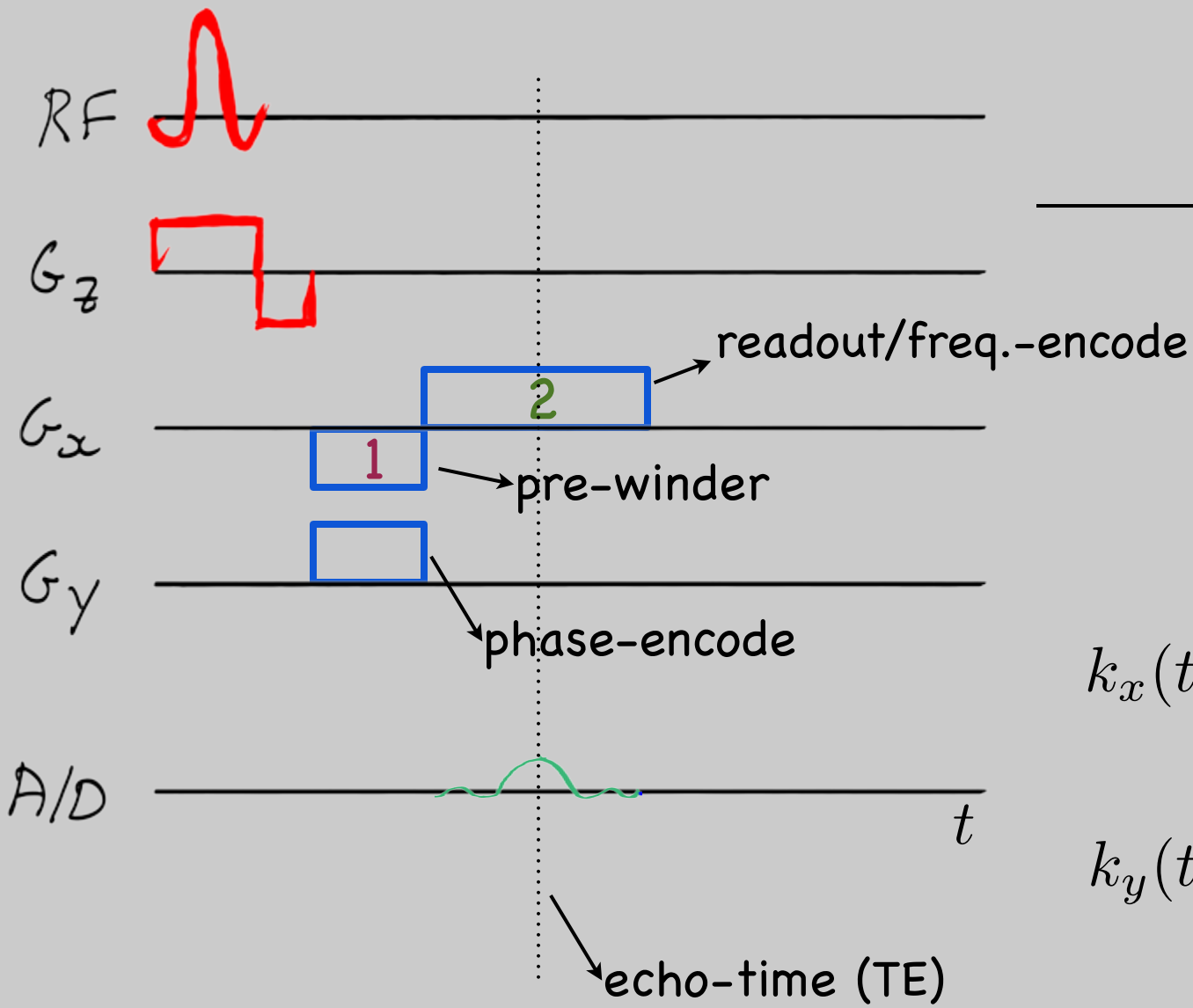
2DFT (Spin Warp)

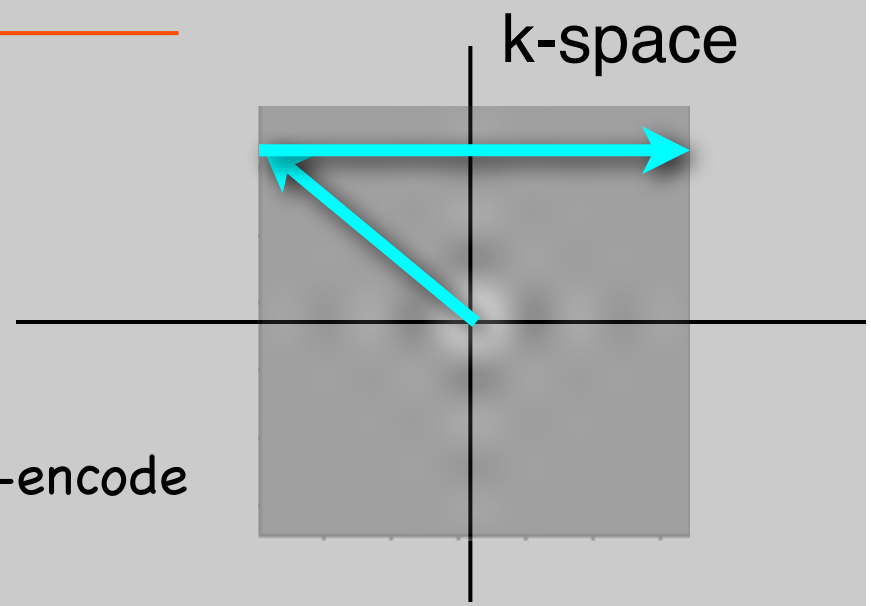


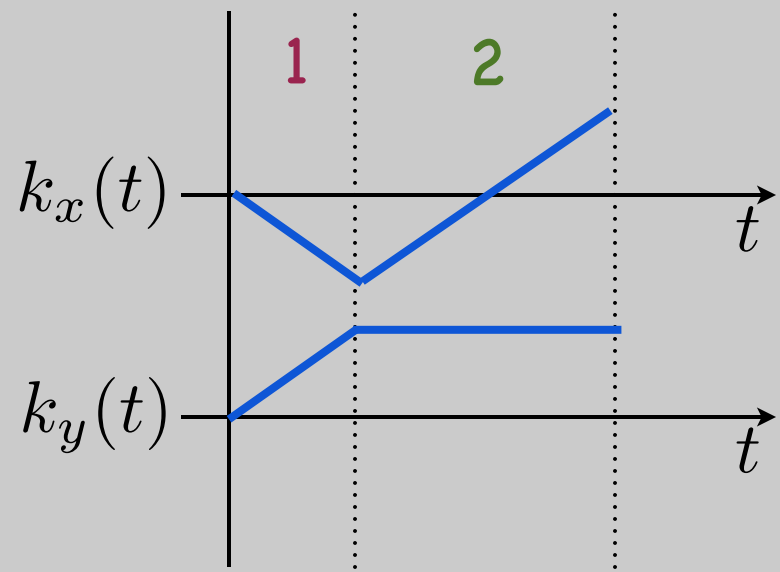




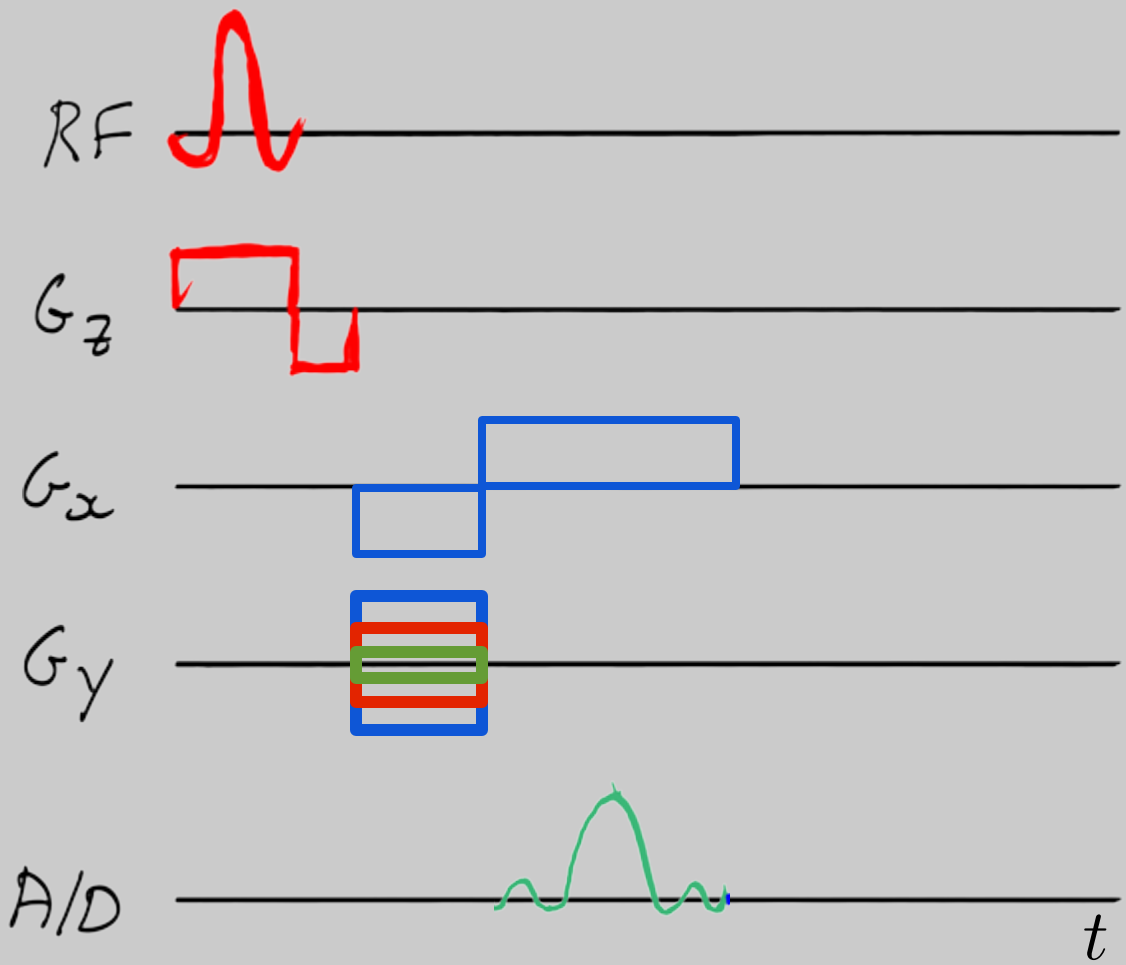
2DFT (Spin Warp)

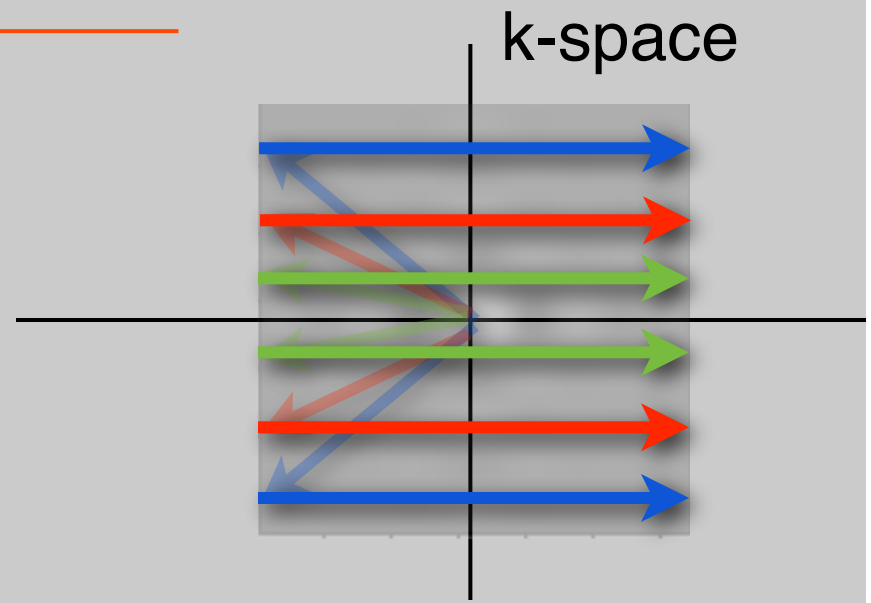






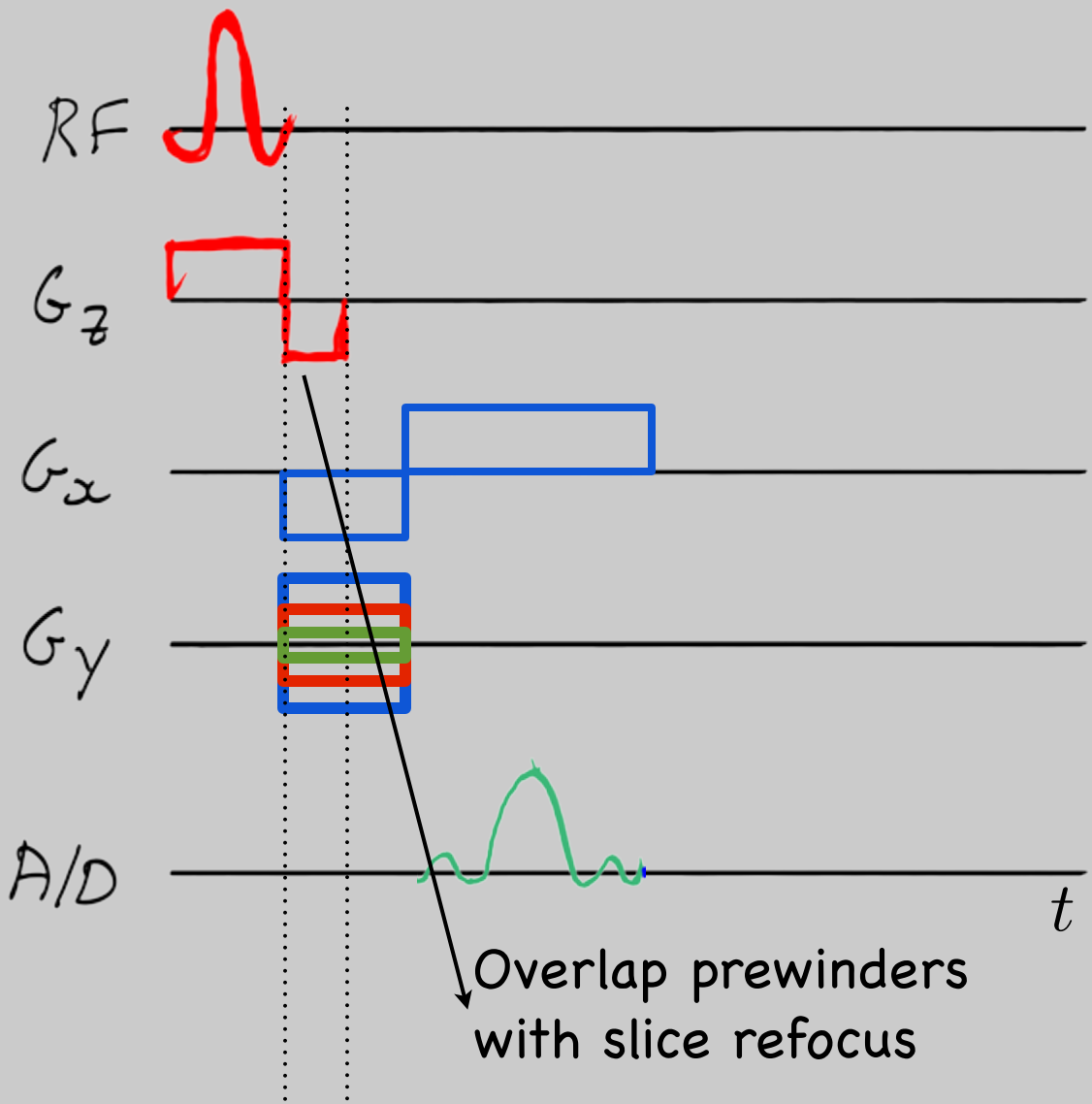
2DFT (Spin Warp)

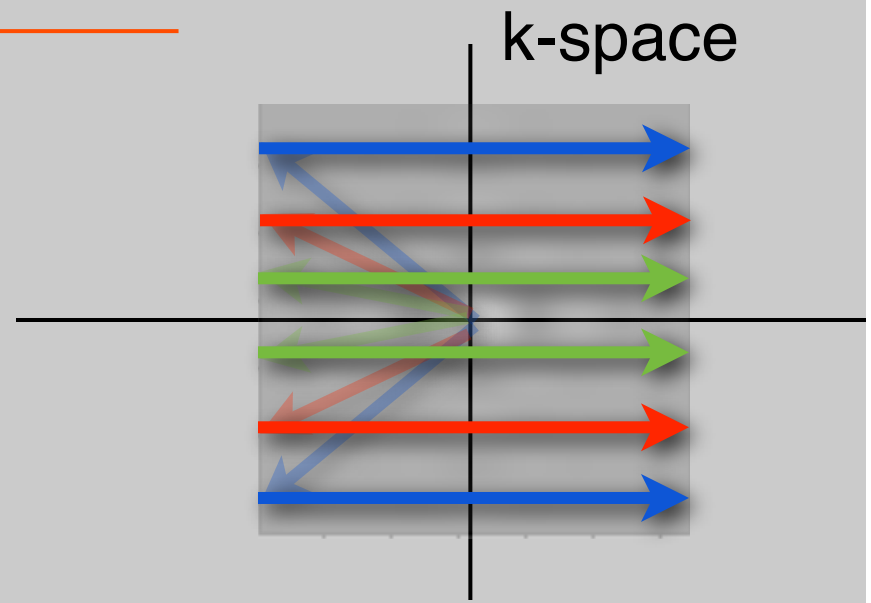




- Repeat for each line
- Obtain 2D Cartesian Grid
- By far the most common trajectory!

2DFT (Spin Warp)





- Repeat for each line
- Obtain 2D Cartesian Grid
- By far the most common trajectory!

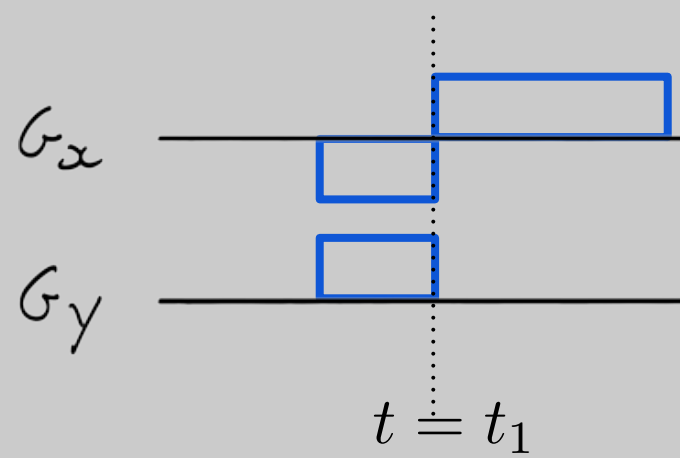
2DFT (Spin Warp)

- Gy gradient is Phase-Encode gradient
  - Establishes fixed linear-phase before readout

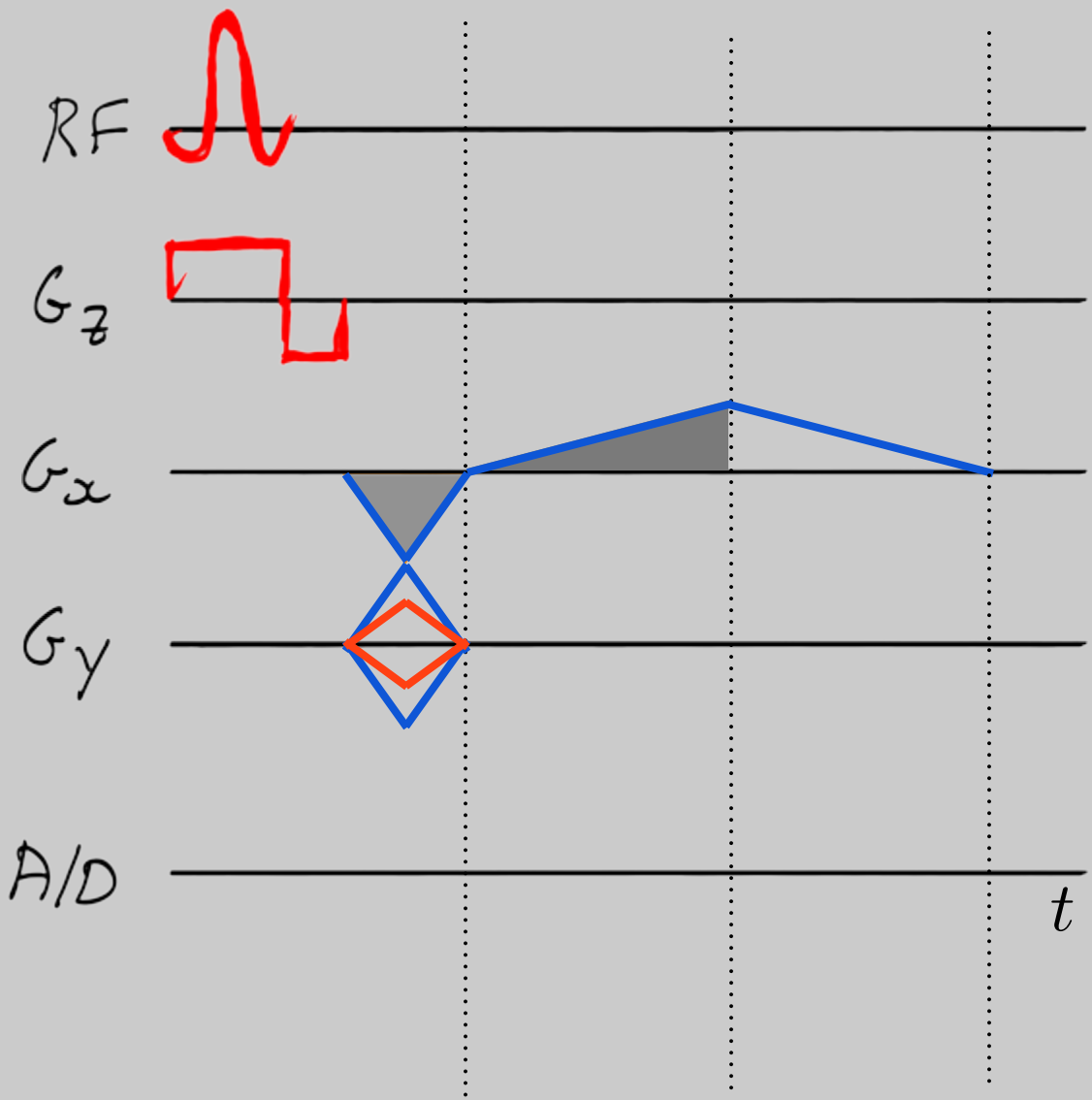
$$M_{xy}(\vec{r},0)e^{-i2\pi k_y(t_1)y}$$

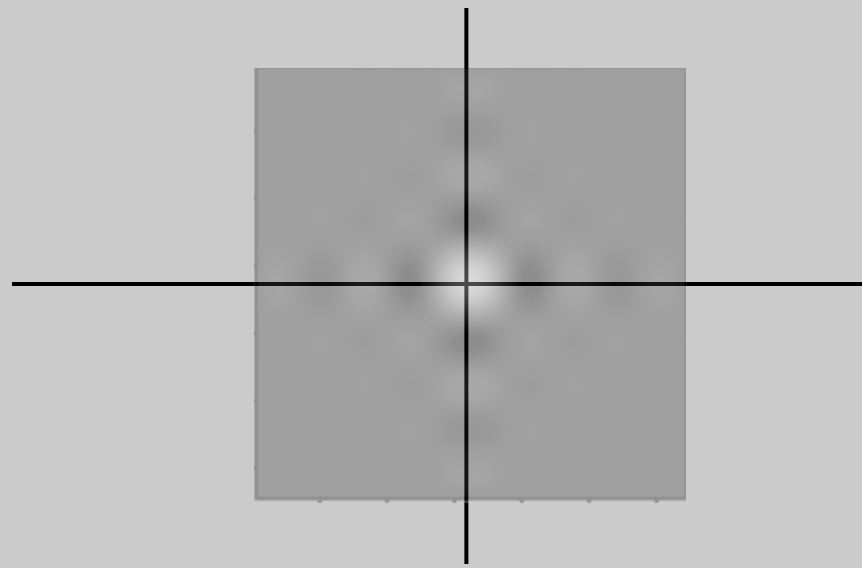
- Gx gradient is Frequency-Encode gradient, or Readout

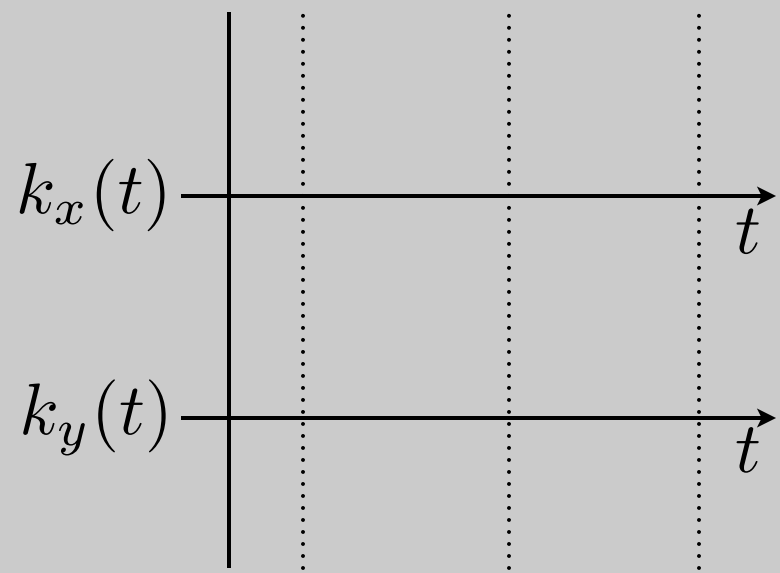
$$M_{xy}(\vec{r},0)\underbrace{e^{-i2\pi k_y(t_1)y}}_{\text{fixed}} \cdot e^{-i2\pi k_x(t)x}$$



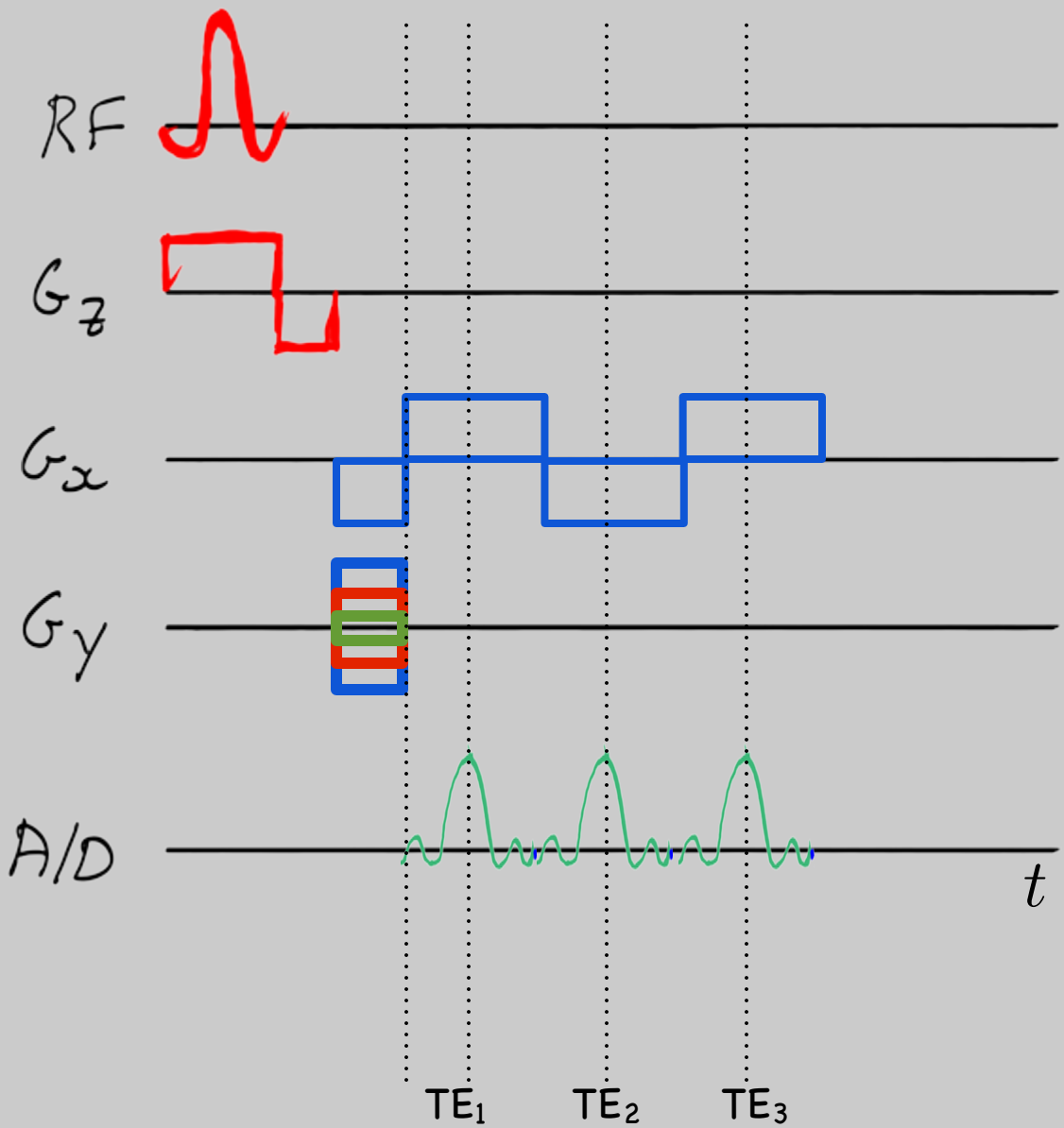
What Does This Sequence Do?



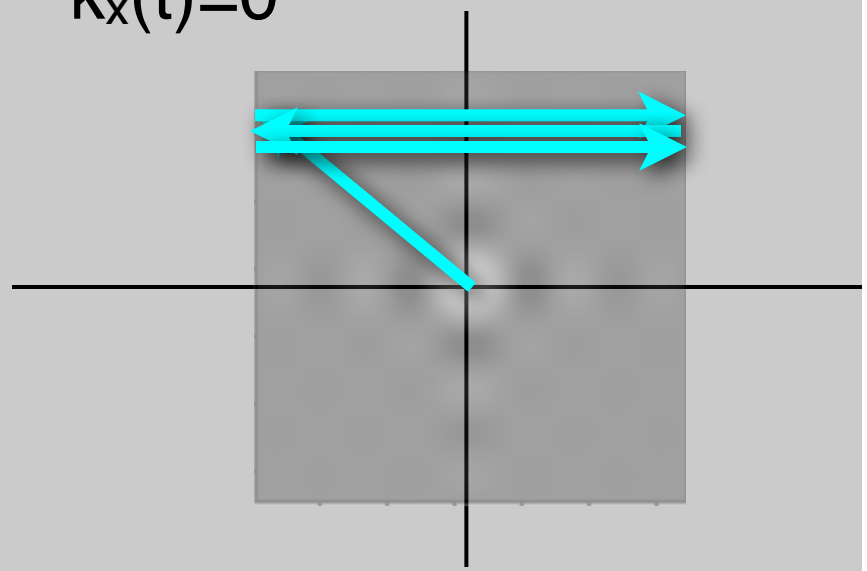


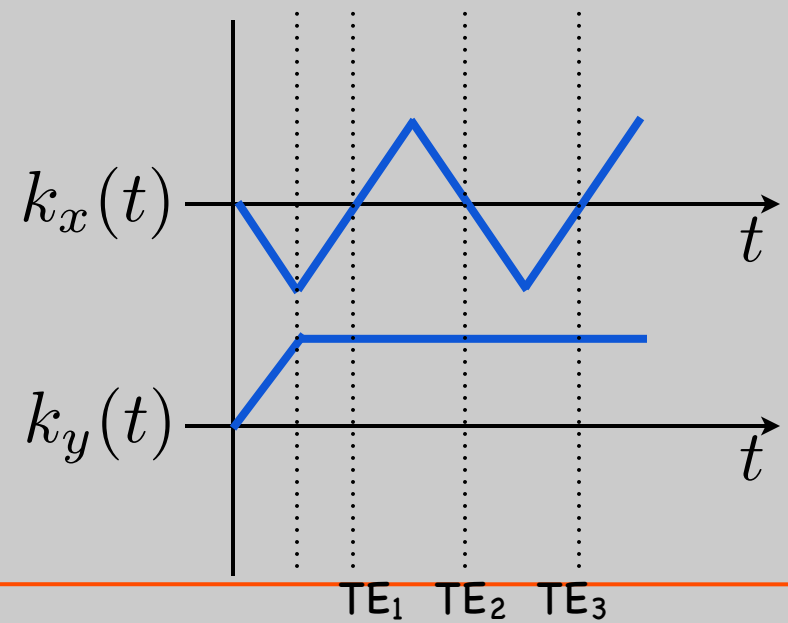


Multi-Echo 2DFT

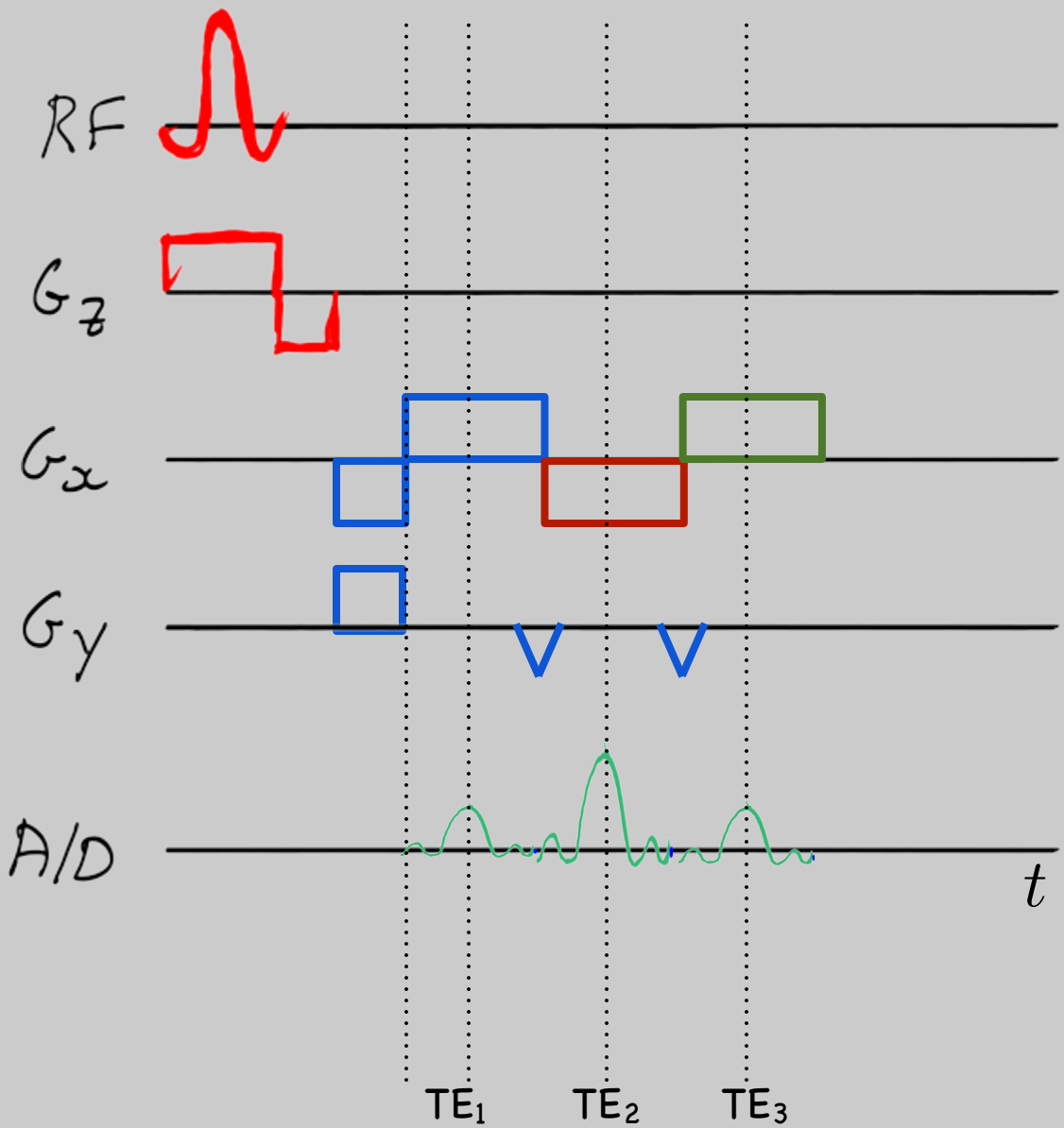


- Echo-Time is when $k_x(t)=0$

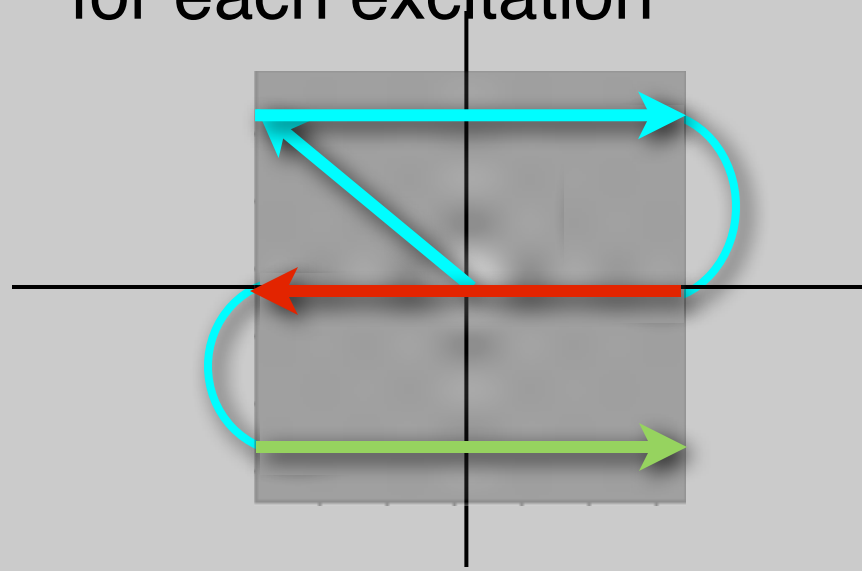


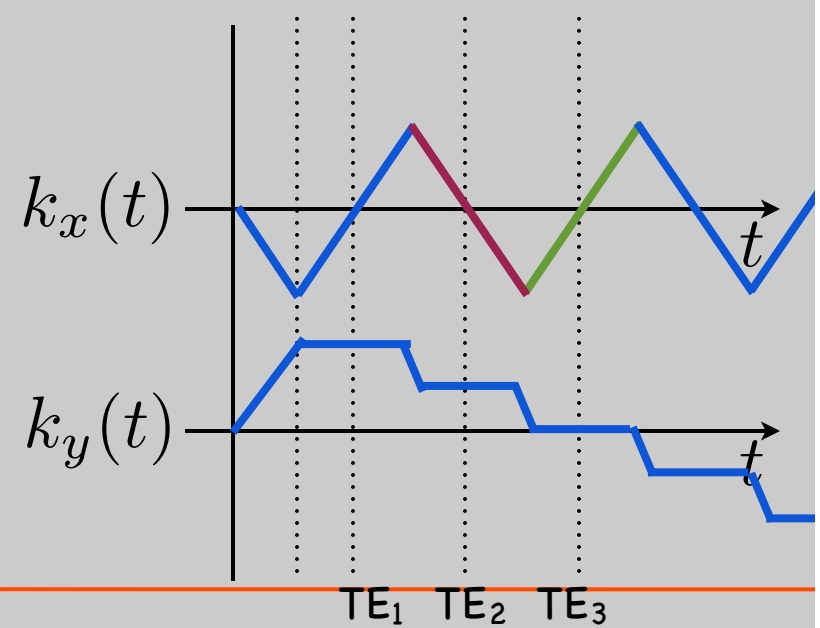


Echo Planar Imaging (EPI)

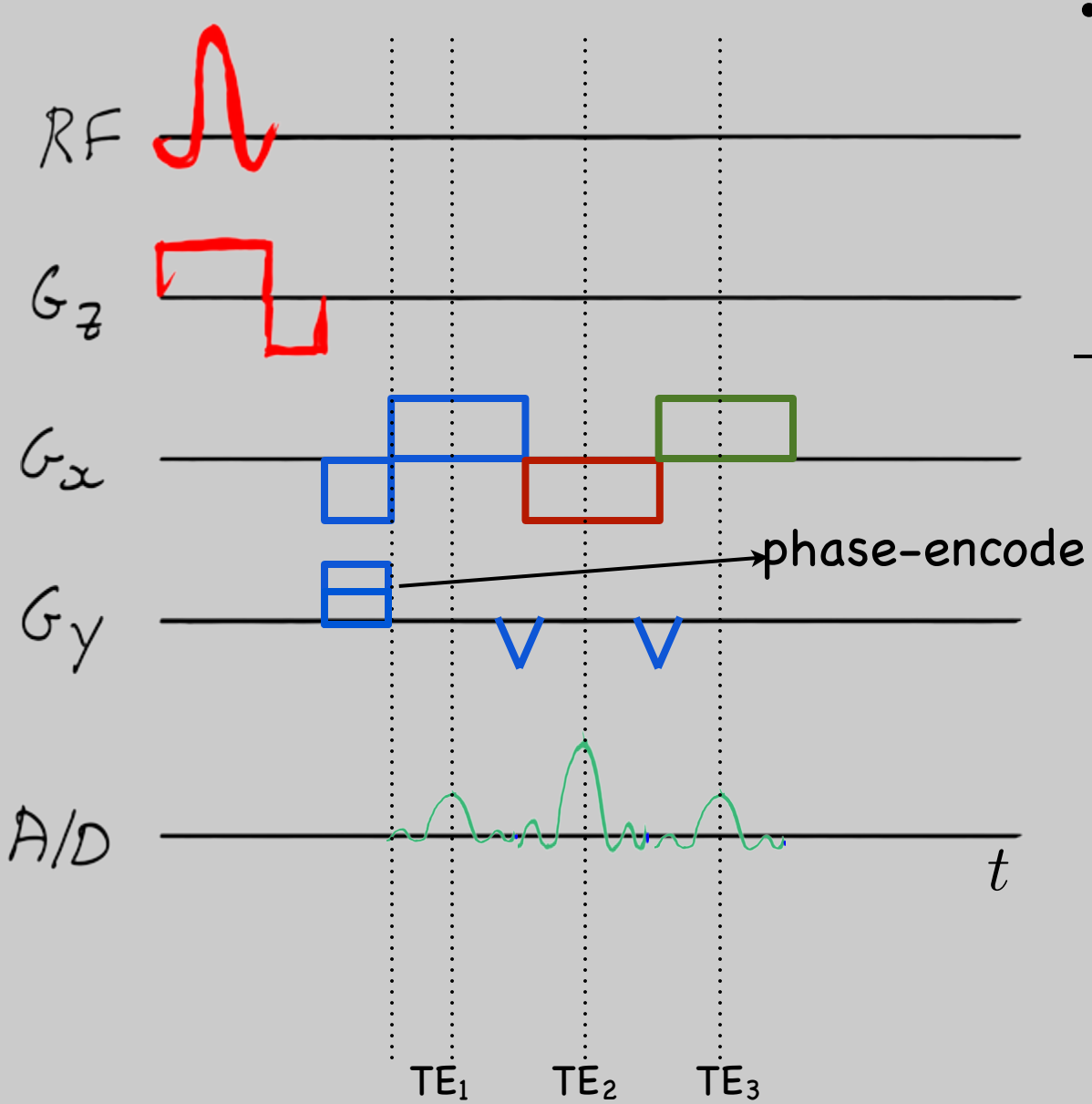


- More phase encode for each excitation

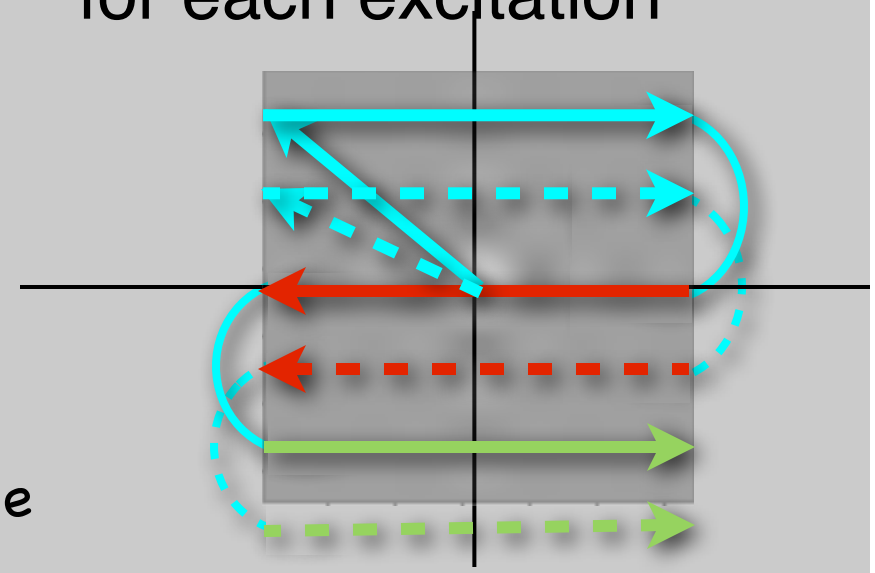


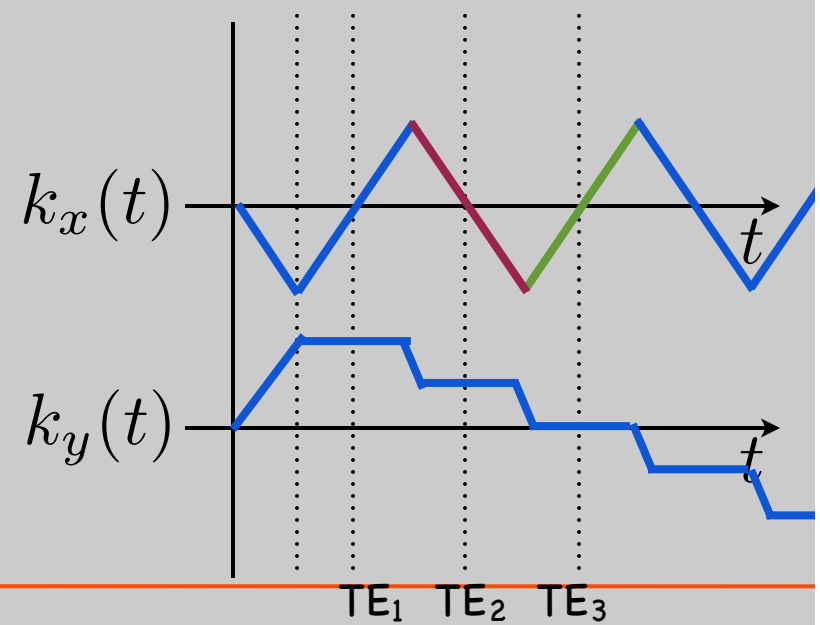


Echo Planar Imaging (EPI)

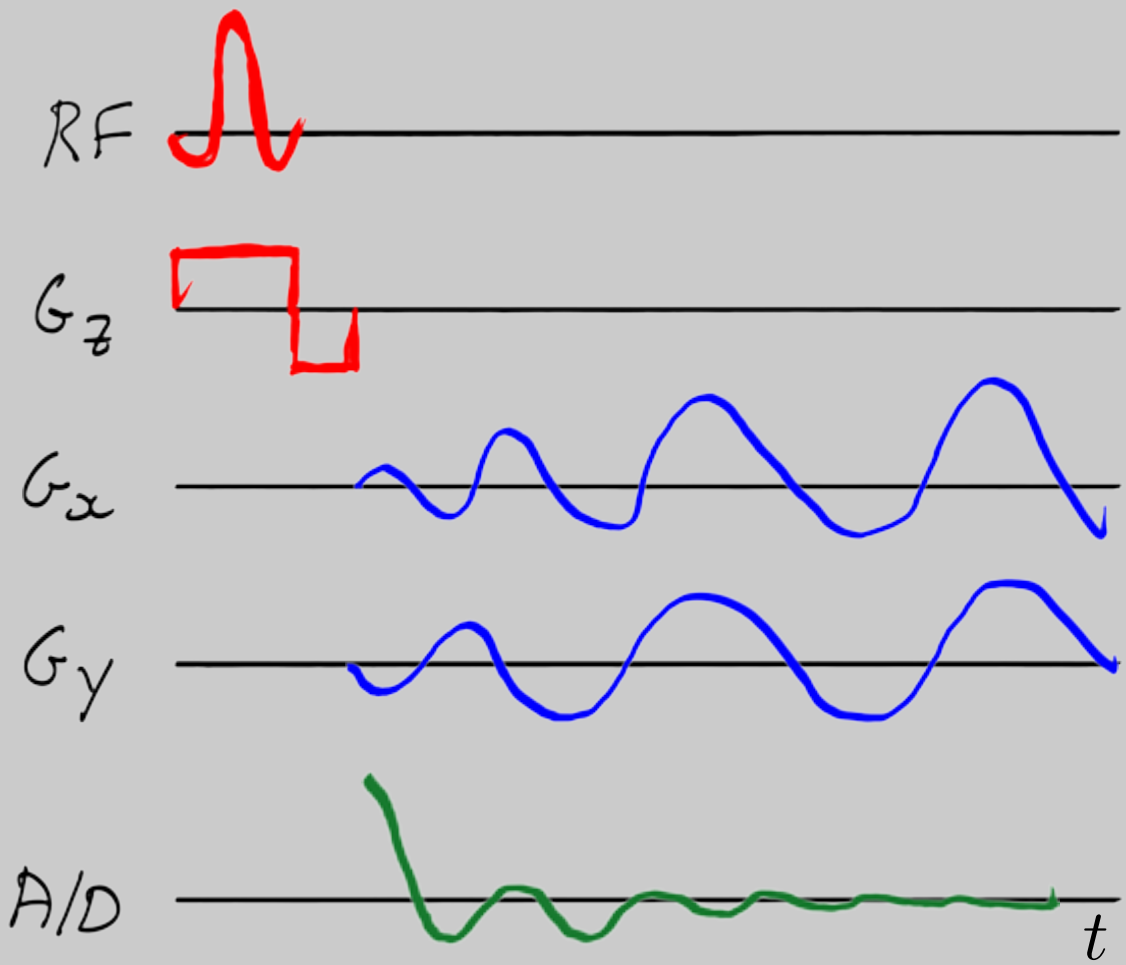


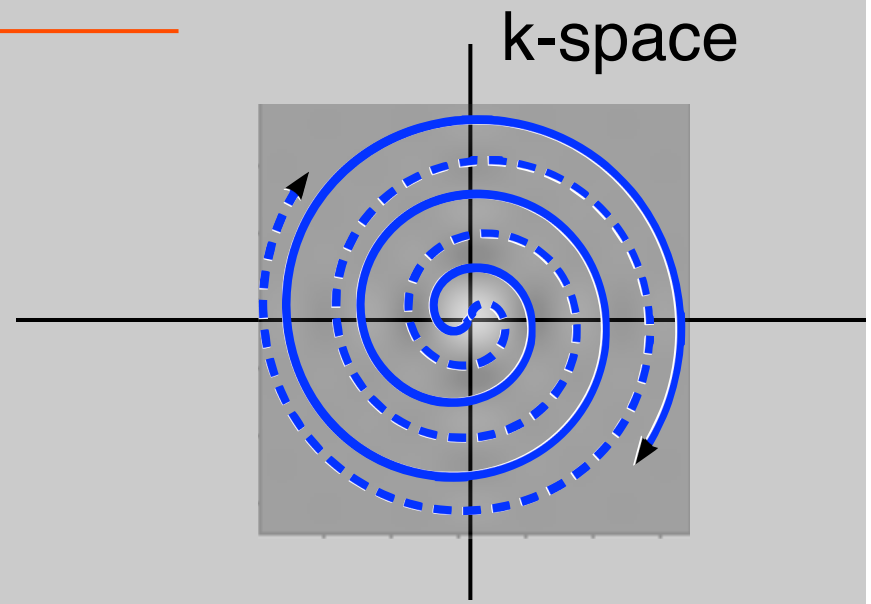
- More phase encode for each excitation

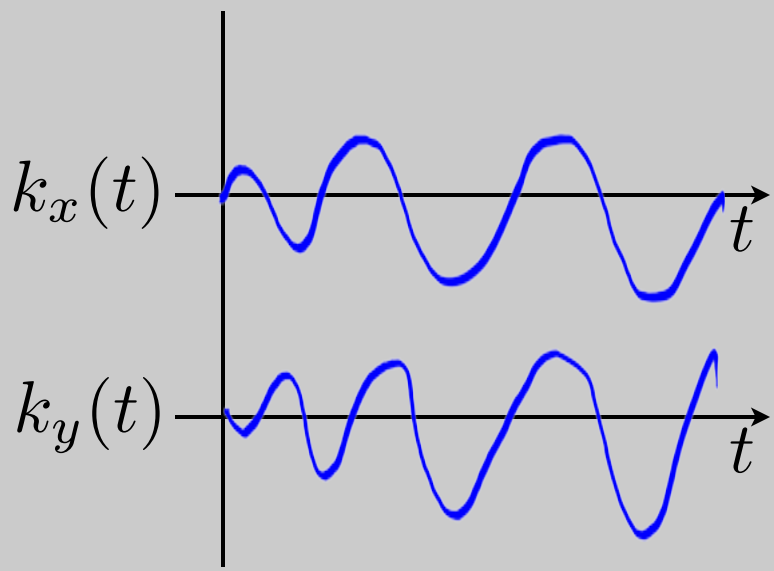




Spiral







EPI and Spiral

- EPI: Most common high-speed acq.
  - fMRI, diffusion, real-time
  - Can use interleaved multiple shots
  - “Easy” DFT reconstruction

- Spiral
  - Very hardware efficient
  - Useful for flow/ hearts, also used for fMRI
  - Often multiple interleaved shots
  - Uses gridding for non-uniform FT reconstruction

Sampling, Resolution and FOV

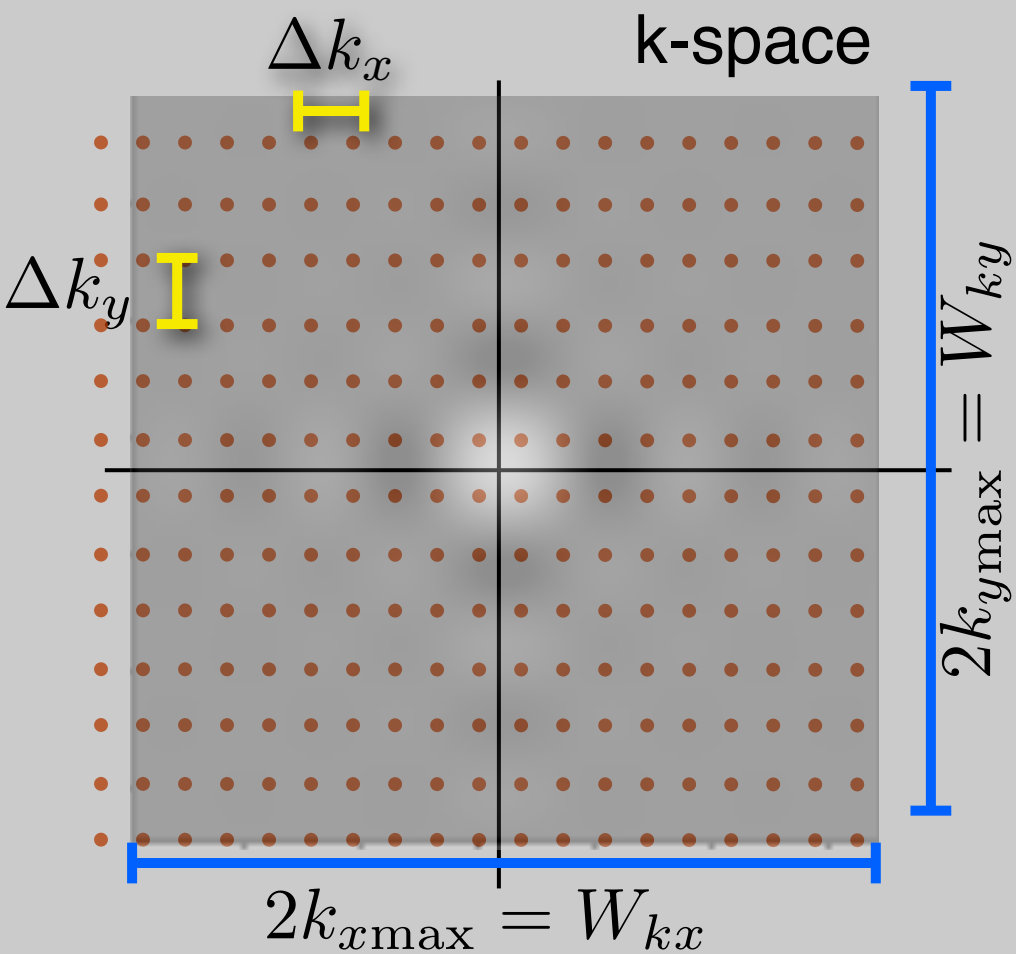
- Signal is sample of FT of object

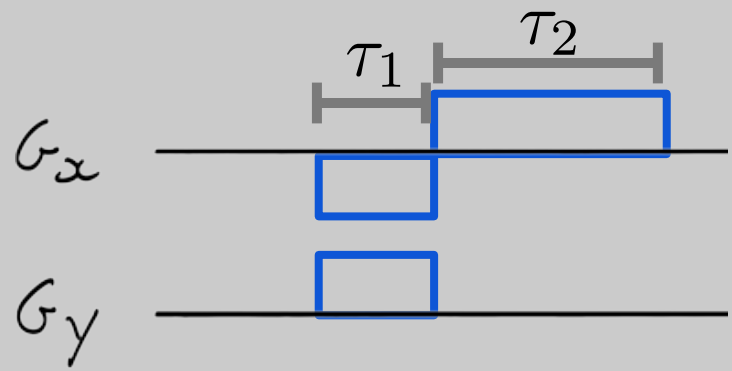
$$s(t) = \tilde{\mathcal{F}}\left\{ M_{xy}(\vec{r}, 0) \right\} \Big|_{\vec{k} = \vec{k}(t)}$$

- sampled at discrete times!

$$s(n\Delta t) = m_{xy}(\vec{k}(n\Delta t))$$

2DFT Case





- Nr - # readout samples
- Np - # phase encodes
  - $k_{y\max} = \frac{\gamma}{2\pi} G_y \tau_1$
  - $W_{kx} = \frac{\gamma}{2\pi} G_x \tau_2$
  - $\Delta k_x = \frac{W_{kx}}{N_r}$
  - $\Delta k_y = \frac{W_{ky}}{N_p}$

Sampled Fourier Transform

- In k-space

$$\hat{m}_{xy}(k_x,k_y) = \underbrace{m(k_x,k_y)}_{\text{F.T of object}} \underbrace{\left[\frac{1}{\Delta k_x \Delta k_y} \overset{2}{\sqcup\!\sqcup}\left(\frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y}\right)\right]}_{\text{Sampling}} \cdot \underbrace{\overset{2}{\Pi}\left(\frac{k_x}{W_{k_x}}, \frac{k_y}{W_{k_y}}\right)}_{\text{Extent}}$$

- In Image space

$$\hat{M}_{xy}(x,y) = \underbrace{M_{xy}(x,y)}_{\text{cont. IMAGE}} ** \underbrace{\overset{2}{\sqcup\!\sqcup}(\Delta k_x x, \Delta k_y y)}_{\text{Replication/ Aliasing}} ** W_{k_x} W_{k_y} \underbrace{\operatorname{sinc}(W_{k_x} x)\operatorname{sinc}(W_{k_y} y)}_{\text{BLURRING/ Resolution}}$$

FOV and Aliasing

- Assume infinite resolution (for now...)

$$W_{k_x} \cdot W_{k_y} \operatorname{sinc}(W_{k_x} x) \operatorname{sinc}(W_{k_y} y) \approx {}^{2}\delta(x,y)$$

then,

$$\hat{M}_{xy}(x,y) = M_{xy}(x,y) ** {}^{2}\text{Ш}(\Delta k_x x, \Delta k_y y)$$

- ${}^{2}\text{Ш}(\Delta k_x x, \Delta k_y y)$ has an impulse when:

$$\Delta k x = n \Rightarrow x = \frac{n}{\Delta k_x}$$

$$\Delta k y = m \Rightarrow y = \frac{m}{\Delta k_y}$$

m,n integers

FOV and Aliasing

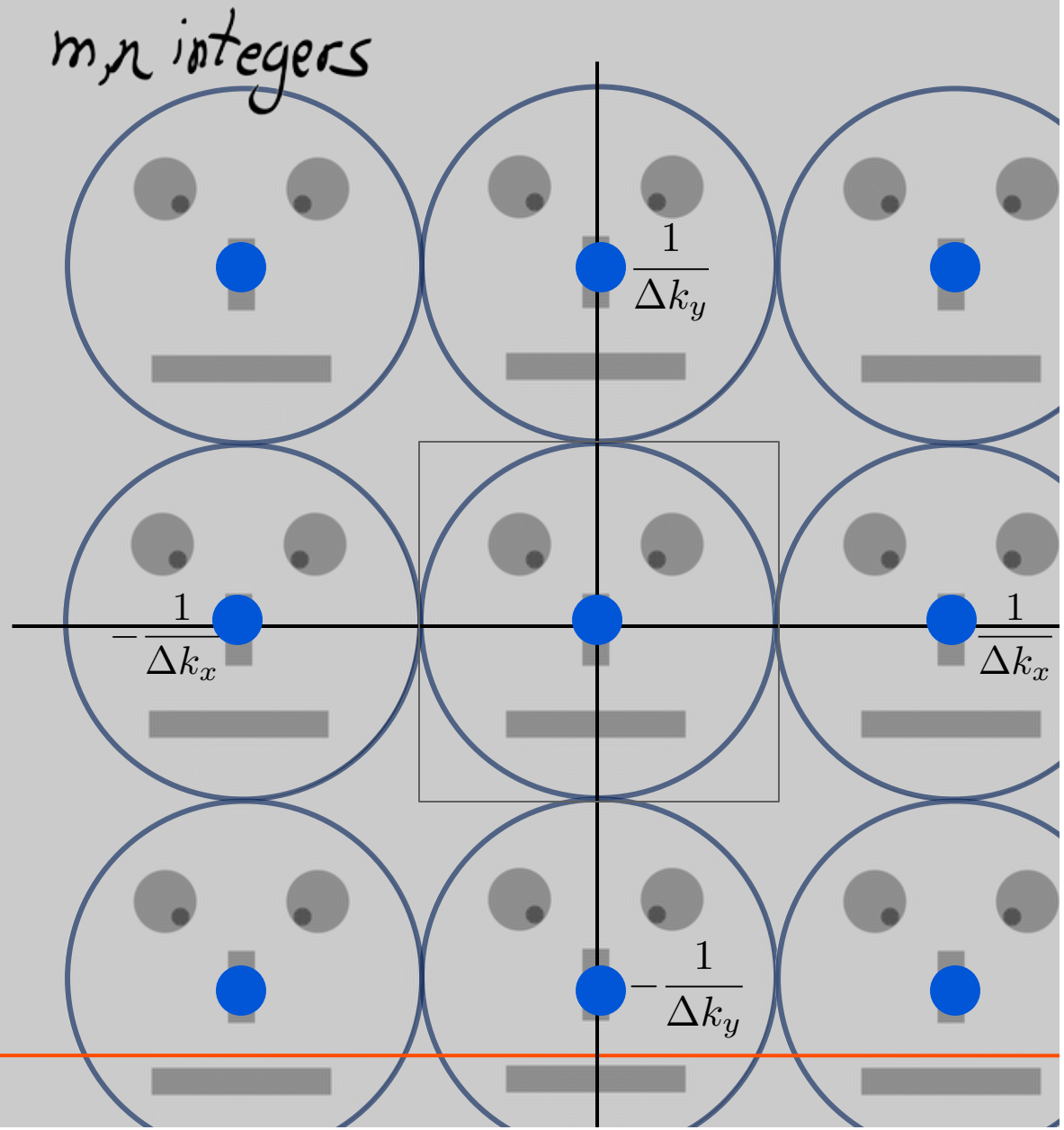
$$\Delta k x = n \Rightarrow x = \frac{n}{\Delta k_x}$$

$$\Delta k y = m \Rightarrow y = \frac{m}{\Delta k_y}$$

$$FOV_x = \frac{1}{\Delta k_x}$$

$$FOV_y = \frac{1}{\Delta k_y}$$

no aliasing as long as object is smaller than FOV



FOV and Aliasing

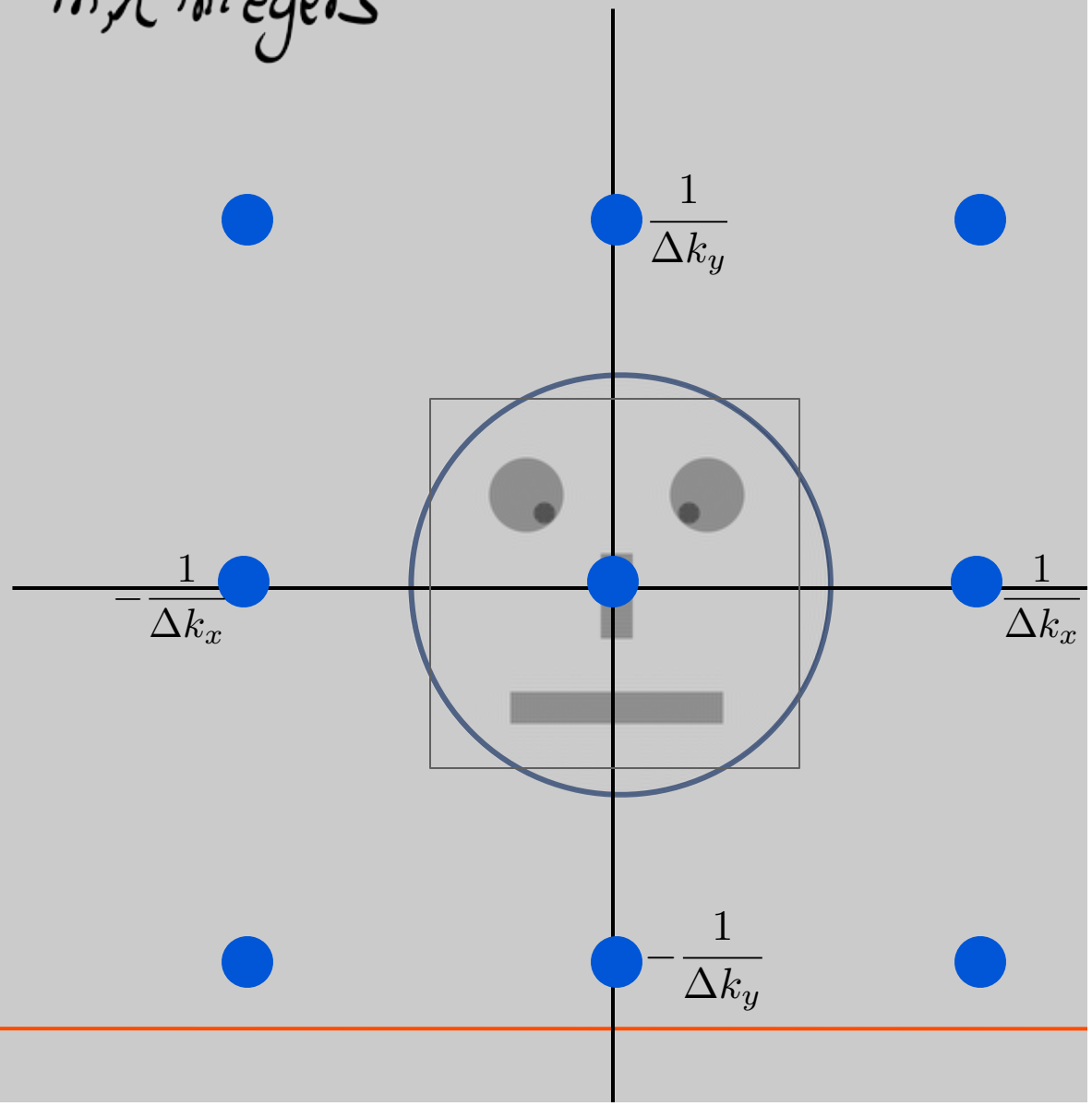
$$\Delta k x = n \Rightarrow x = \frac{n}{\Delta k_x}$$

$$\Delta k y = m \Rightarrow y = \frac{m}{\Delta k_y}$$

m,n integers

$$FOV_x = \frac{1}{\Delta k_x}$$

$$FOV_y = \frac{1}{\Delta k_y}$$



no aliasing as long as object is smaller than FOV

FOV and Aliasing

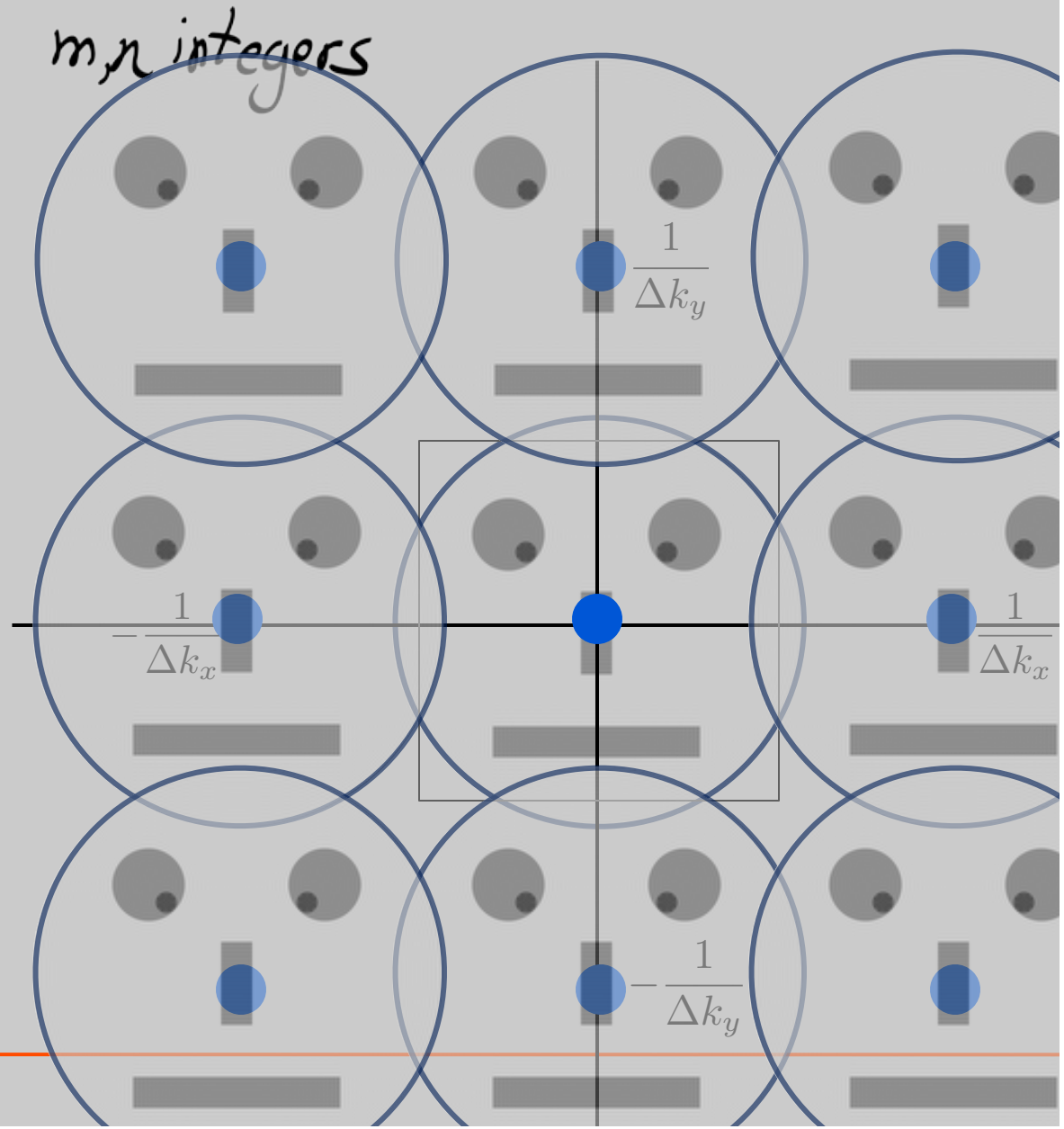
$$\Delta k x = n \Rightarrow x = \frac{n}{\Delta k_x}$$

$$\Delta k y = m \Rightarrow y = \frac{m}{\Delta k_y}$$

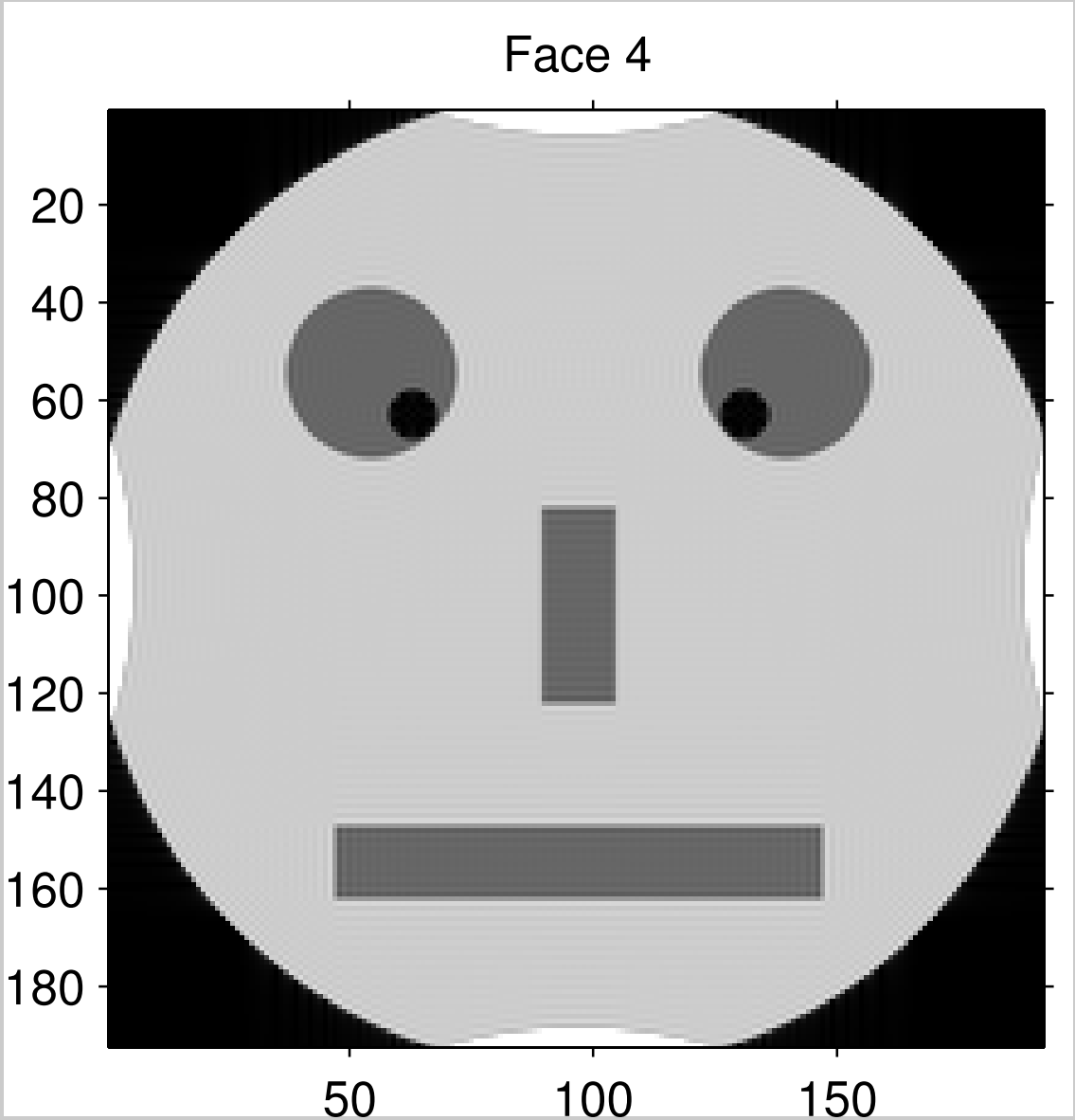
$$FOV_x = \frac{1}{\Delta k_x}$$

$$FOV_y = \frac{1}{\Delta k_y}$$

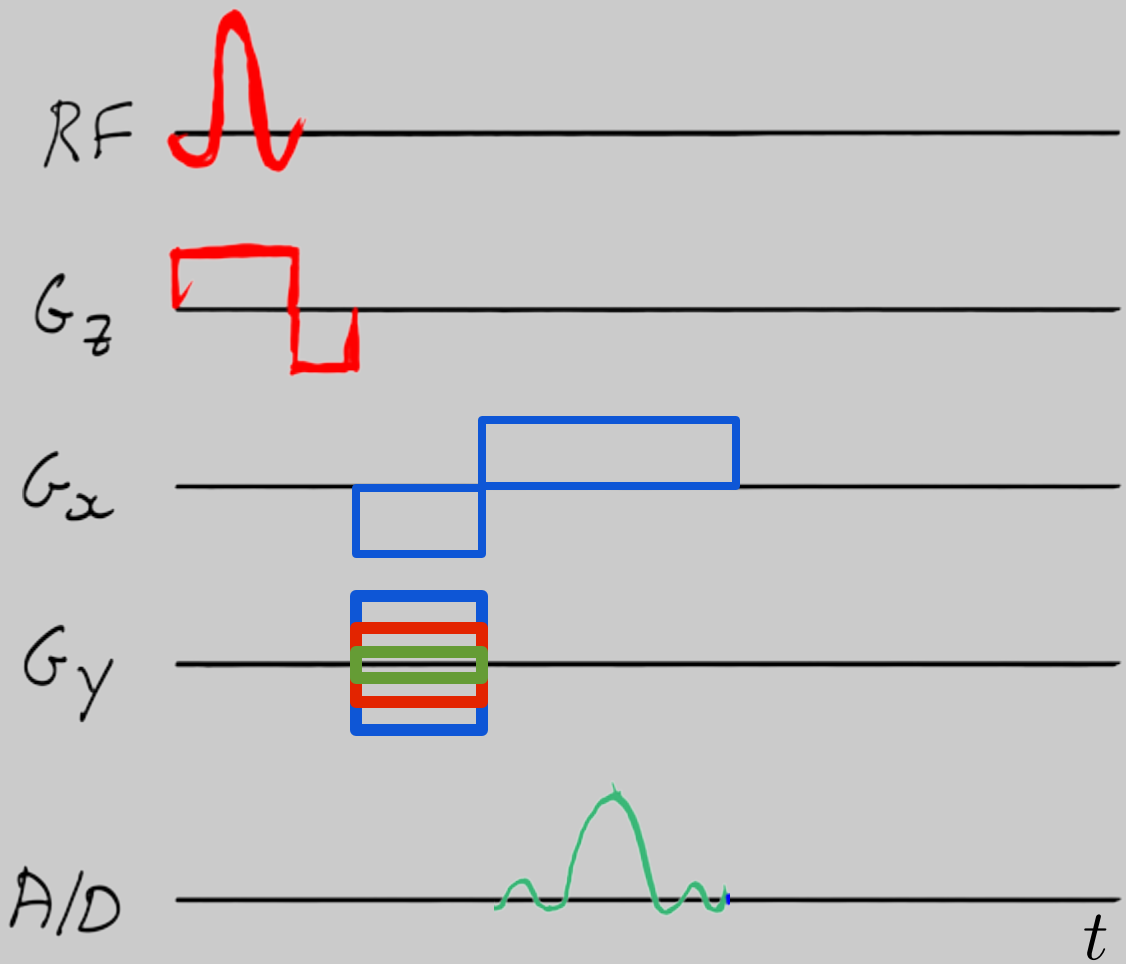
no aliasing as long as object is smaller than FOV

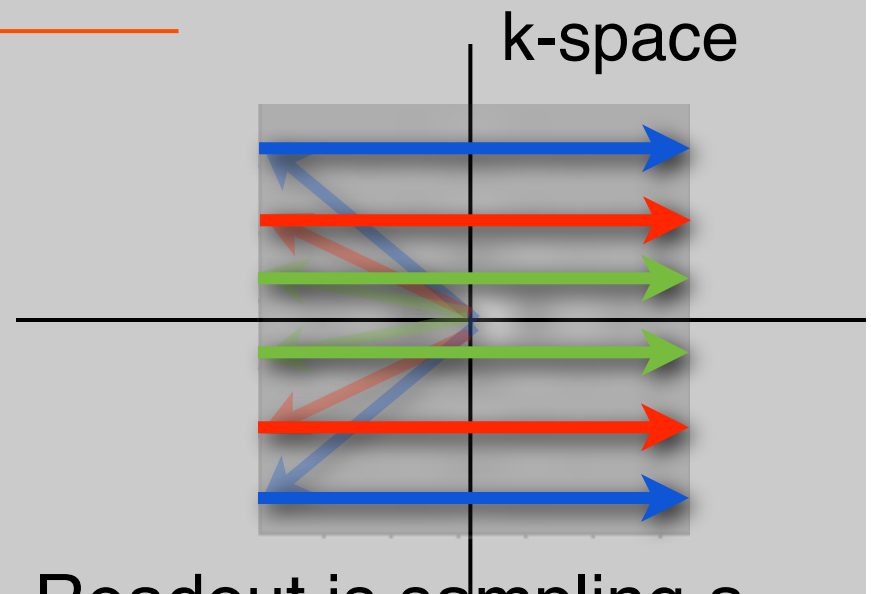


From HW



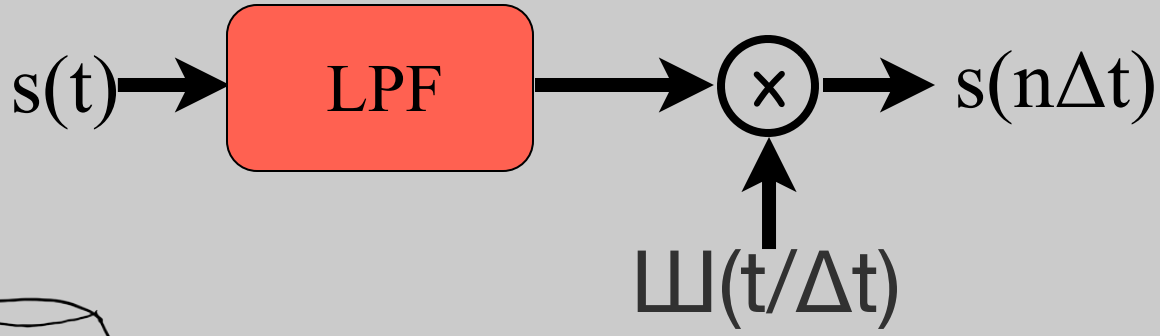
Aliasing in 2DFT

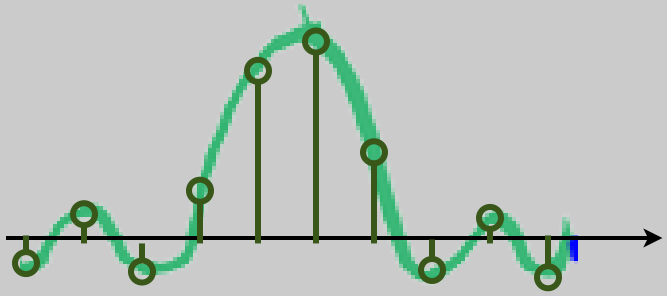


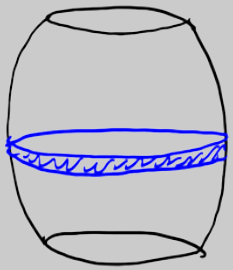


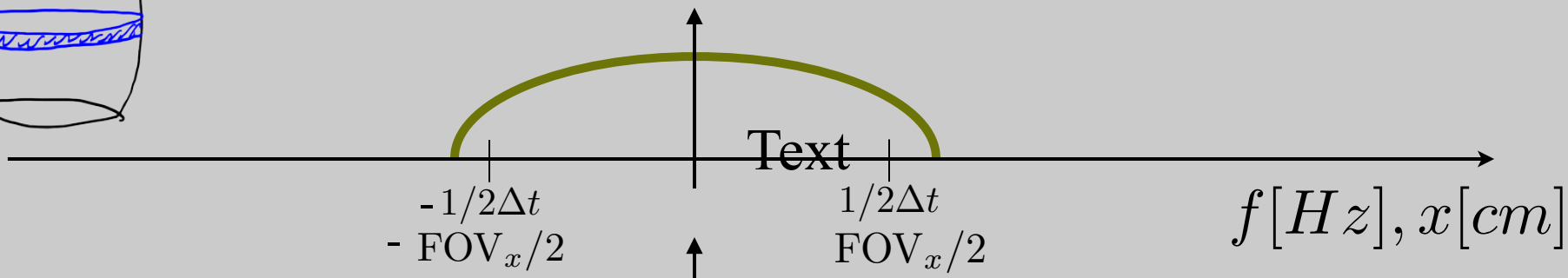
- Readout is sampling a continuous time signal
- Phase encode is inherently discrete!
- Aliasing does not occur in frequency encode direction

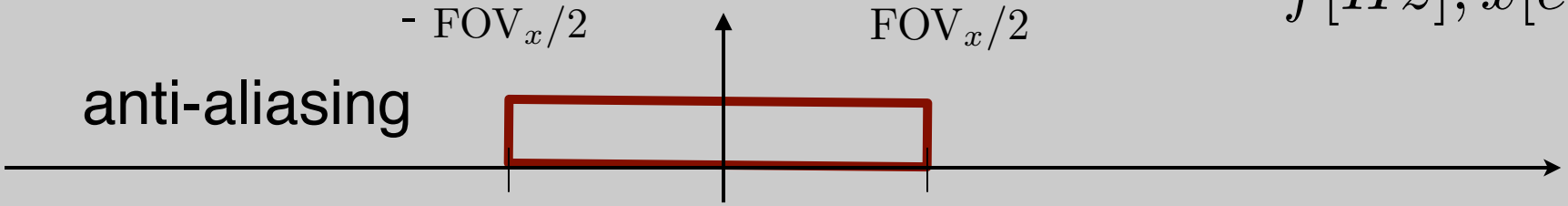
In practice: No Aliasing Along Readout

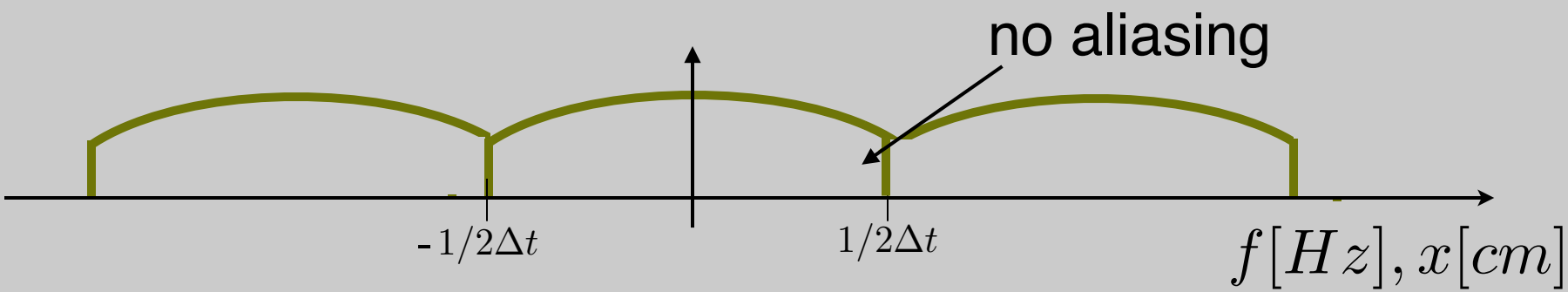








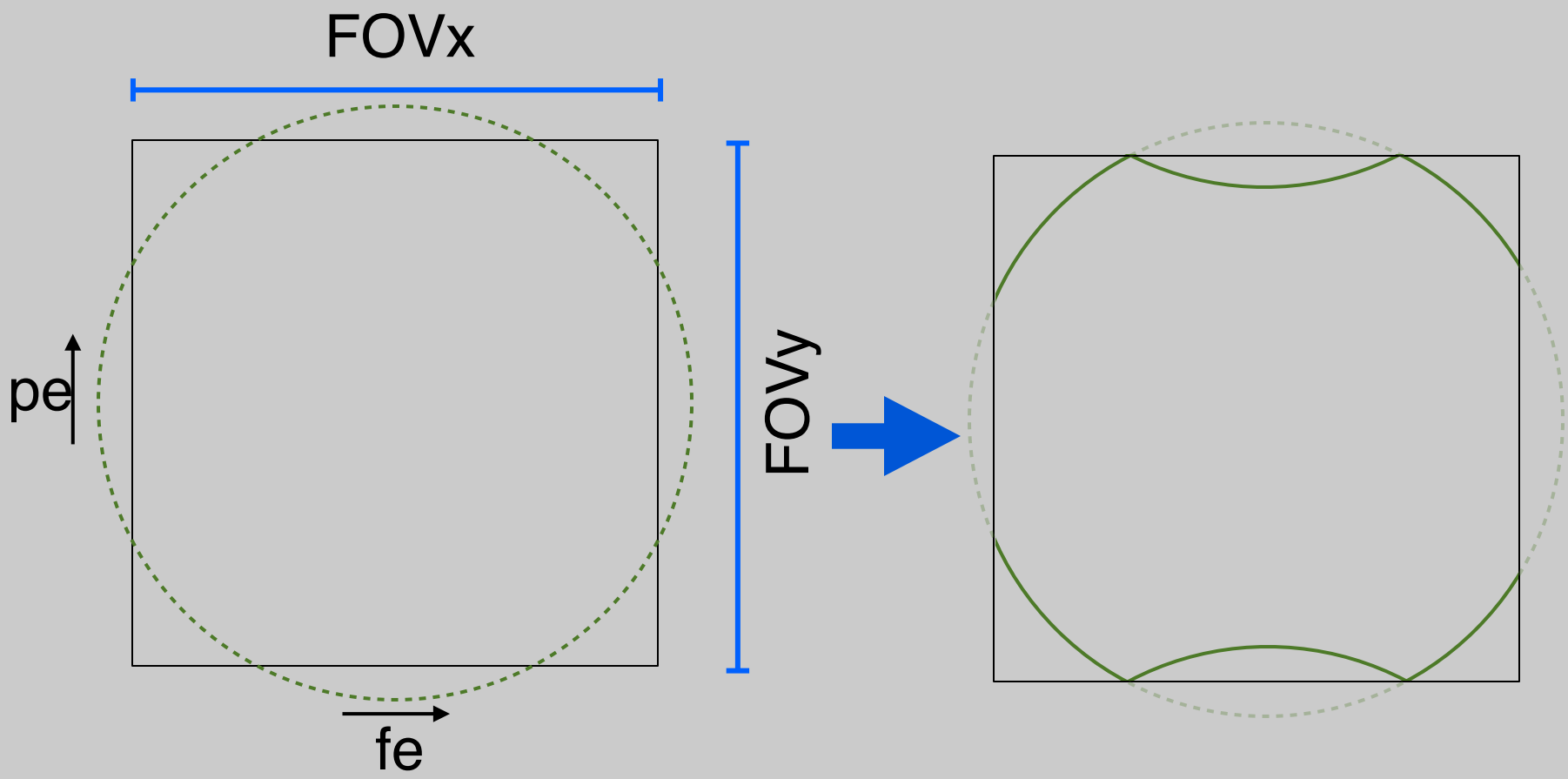




FOV

- In readout/frequency-encode direction
  - Increase FOV by increasing sampling rate
  - No change in acquisition gradients
  - Just more data
  - FOVx is free (almost) → <u>no aliasing</u>
- In phase encode, aliasing can occur
  - Phase-encodes are fundamentally discrete
  - $\mathrm{FOV}_y \Uparrow \quad \Rightarrow \; \Delta k_y = \dfrac{W_{ky}}{N_p} \Rightarrow \; N_p \Uparrow$
  - More acquisitions → more scan time
  - FOV_y costs scan time!

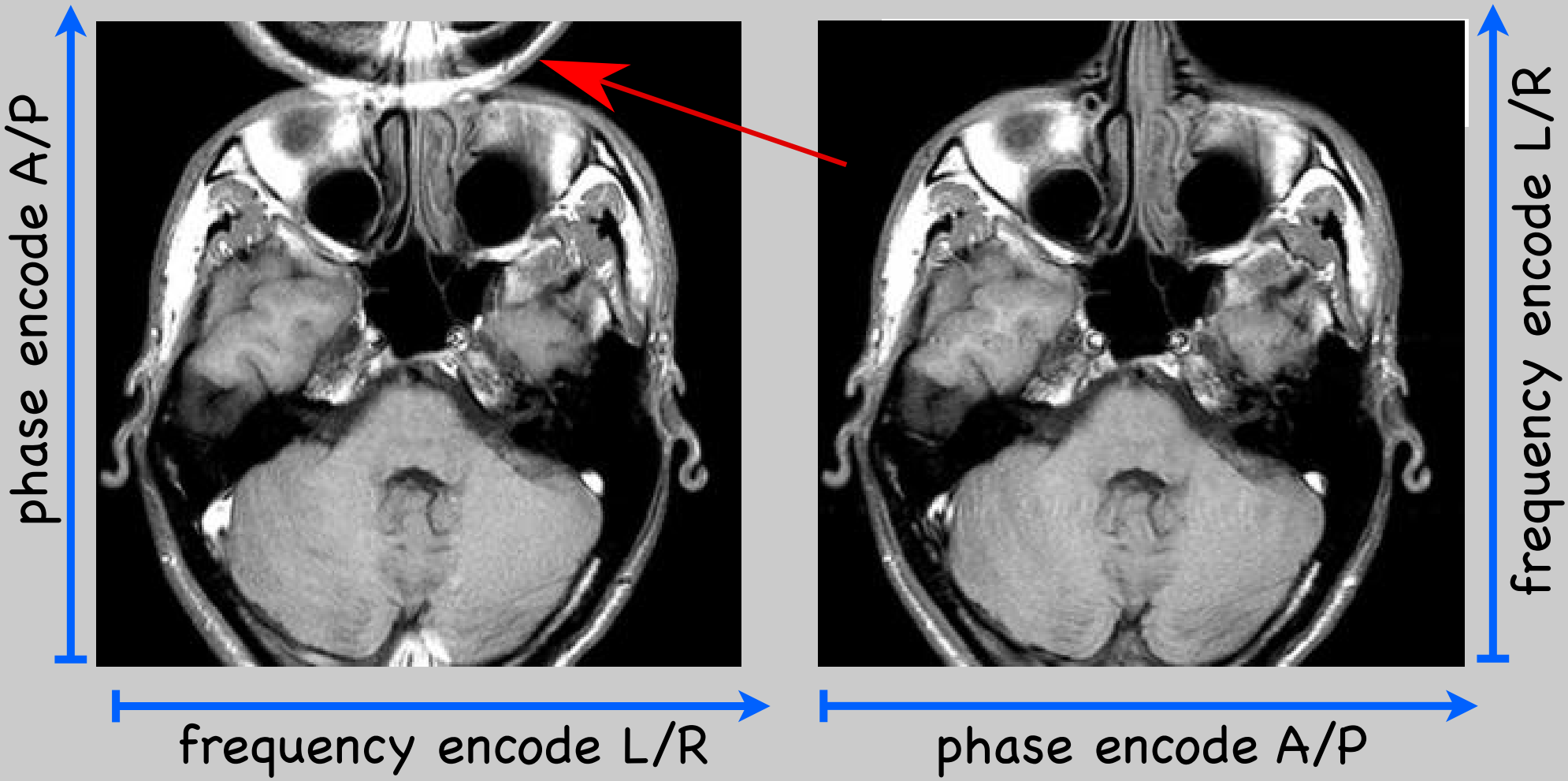
Aliasing



Q: What would the reconstruction image look like?

Aliasing

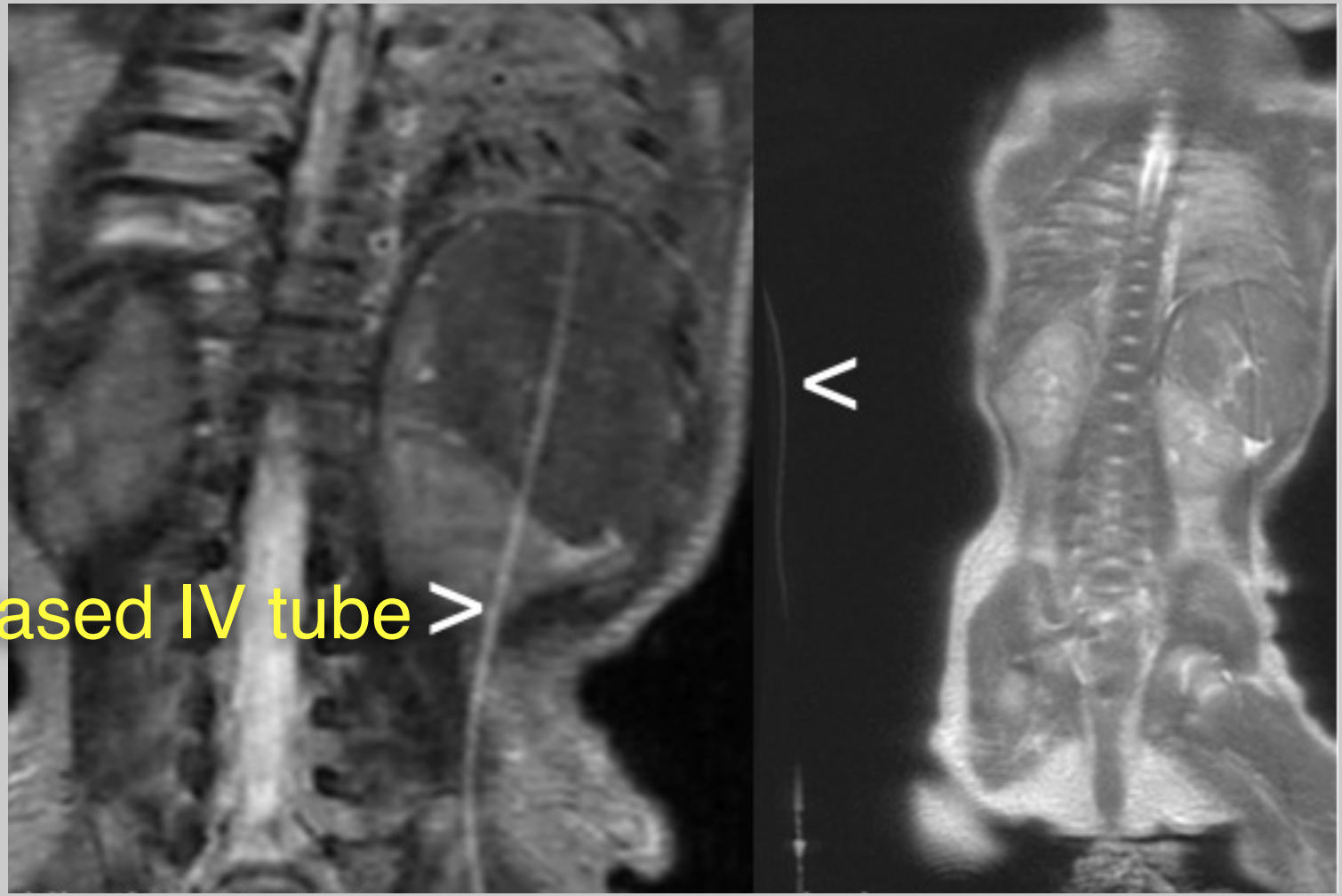
Q: What is the difference in acquisition between the two images ?



Aliasing II

small FOV

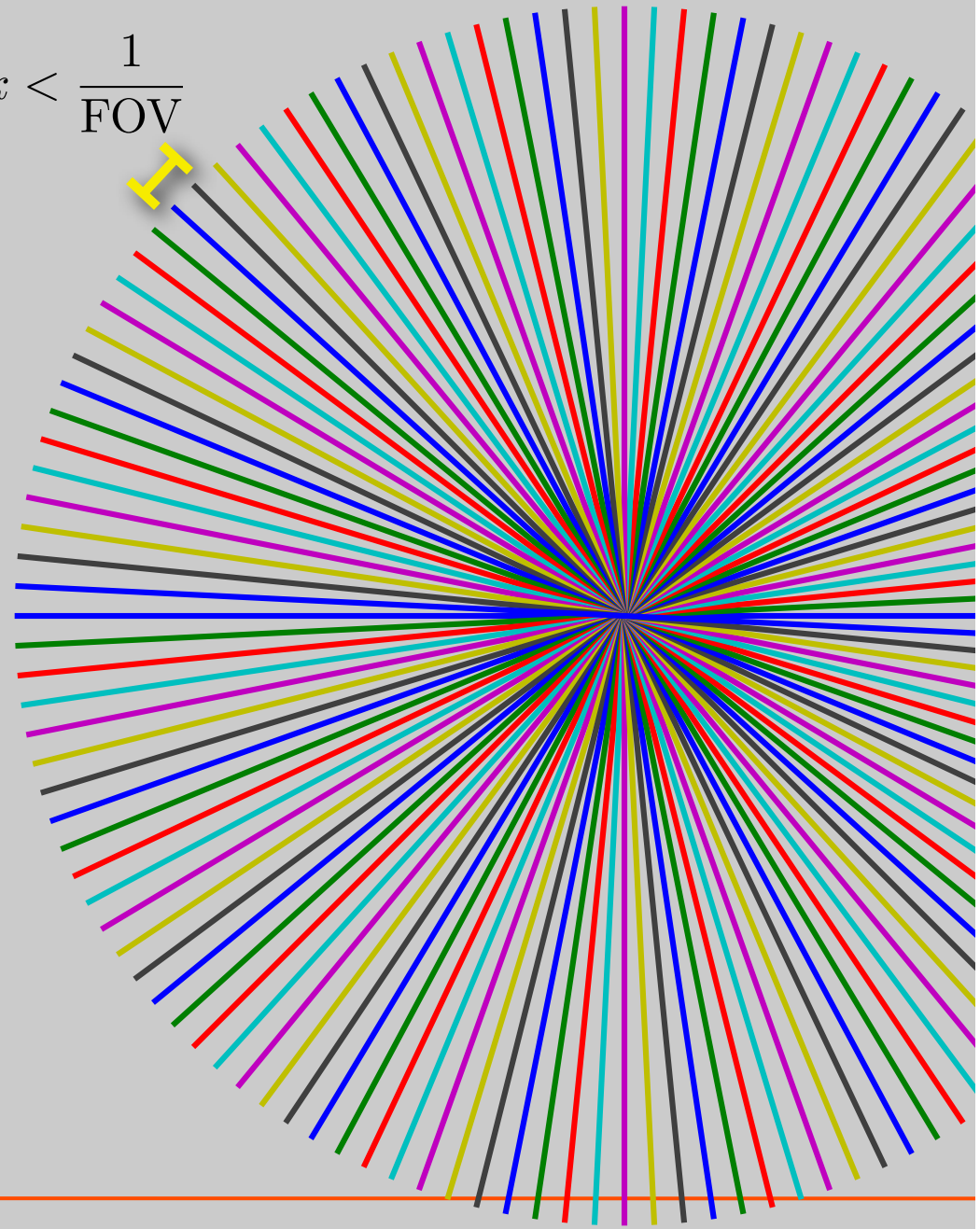
large FOV



Aliasing in Radial Trajectory

- Nyquist criteria still determined by FOV
- Cost π times more than Cartesian
- However:
  - Aliasing is incoherent
  - Degrades gracefully with reduction of spokes

$$\Delta k < \frac{1}{\mathrm{FOV}}$$



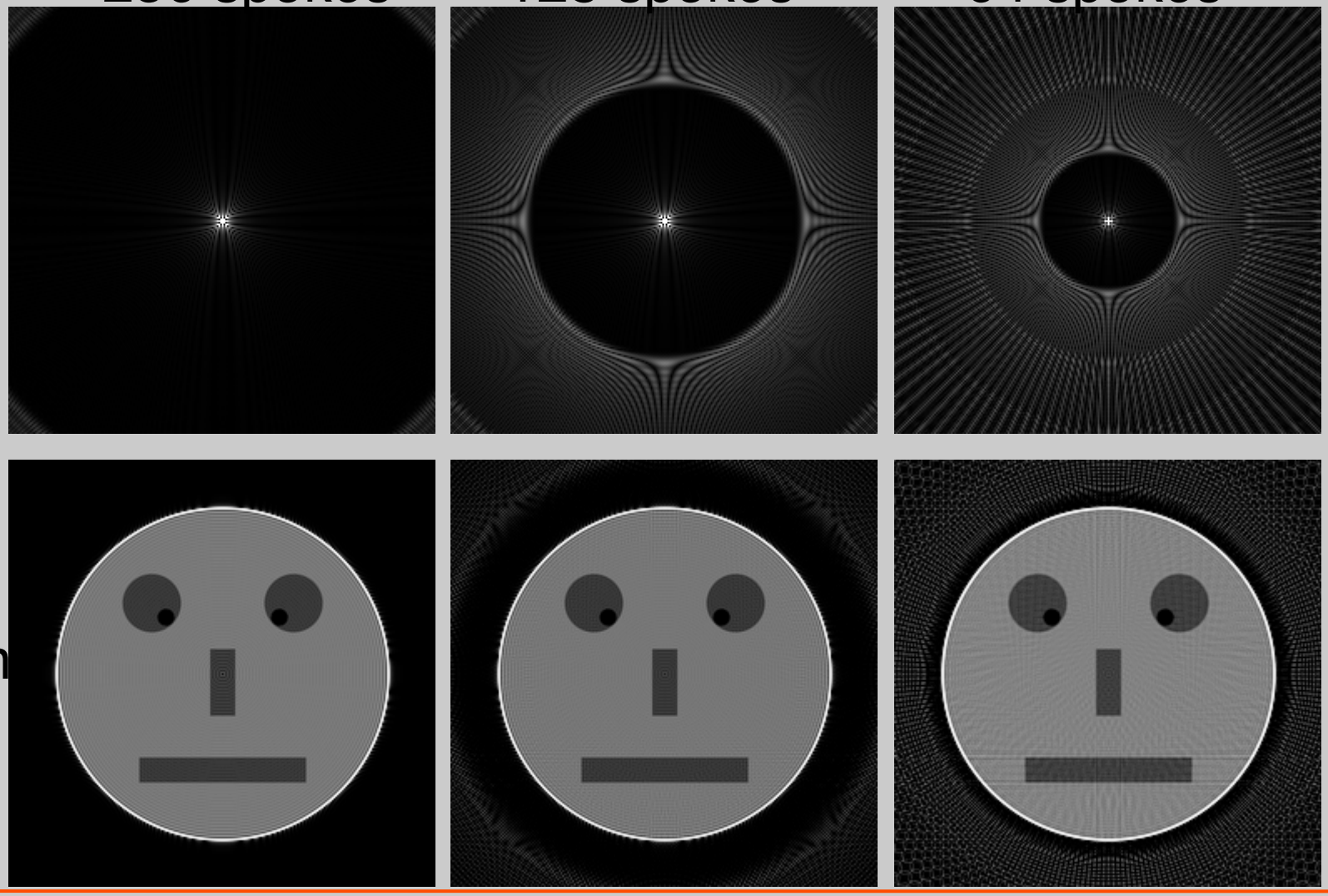
Aliasing in Radial

- 20cm object, 1mm resolution

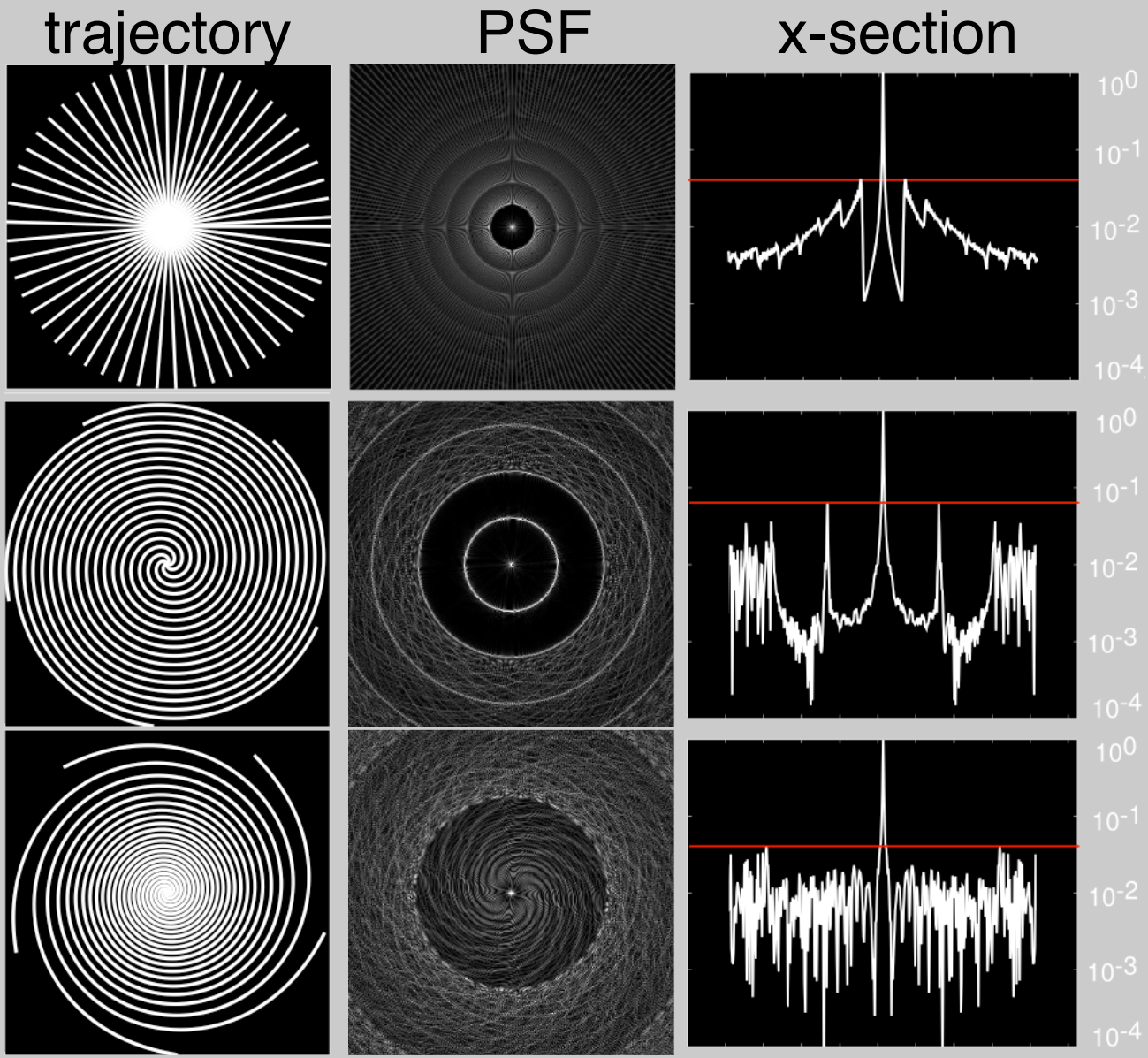
256 spokes | 128 spokes | 64 spokes

Point Spread function:

Reconstuction

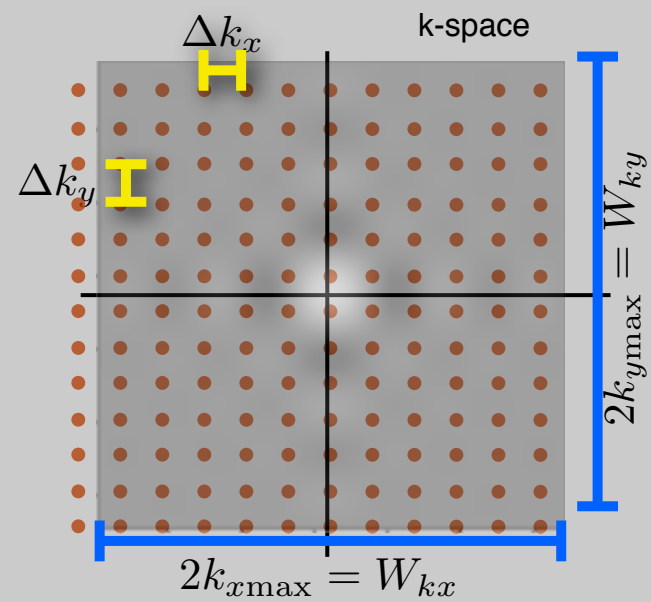


Different Samplings



Spatial Resolution

- The coverage of k-space is finite



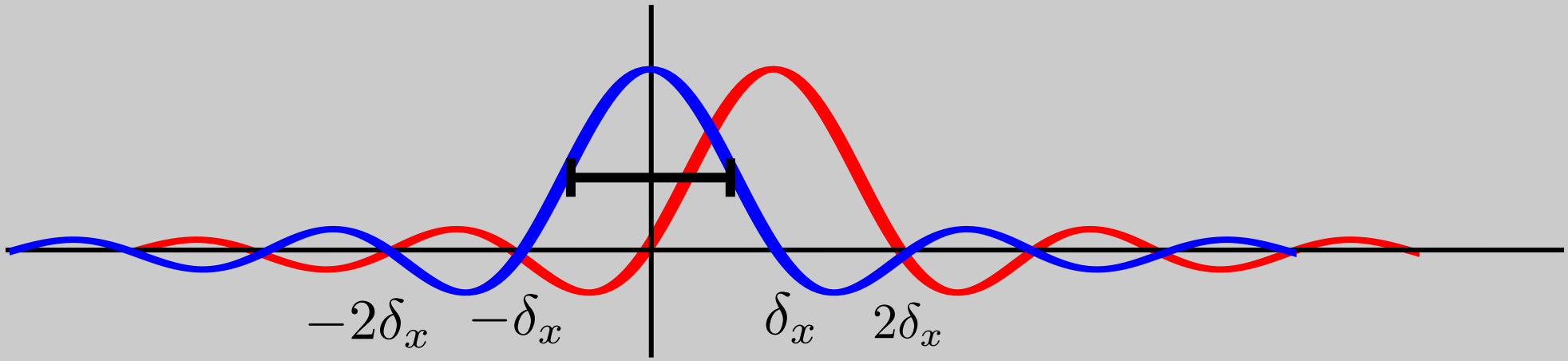
in k-space:

$$\hat{m}_{xy}(k_x,k_y) = \underbrace{m(k_x,k_y)}_{\text{F.T of object}} \underbrace{\left[\frac{1}{\Delta k_x \Delta k_y} \text{Ш}\left(\frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y}\right)\right]}_{\text{Sampling}} \cdot \underbrace{\Pi\left(\frac{k_x}{W_{kx}}, \frac{k_y}{W_{ky}}\right)}_{\text{Extent}}$$

in image space:

$$\hat{M}_{xy}(x,y) = \underbrace{M_{xy}(x,y)}_{\text{cont. IMAGE}} ** \underbrace{\text{Ш}(\Delta k_x x, \Delta k_y y)}_{\text{Replication/Aliasing}} ** W_{kx} W_{ky} \underbrace{\text{sinc}(W_{kx} x)\,\text{sinc}(W_{ky} y)}_{\text{BLURRING/Resolution}}$$

Resolution



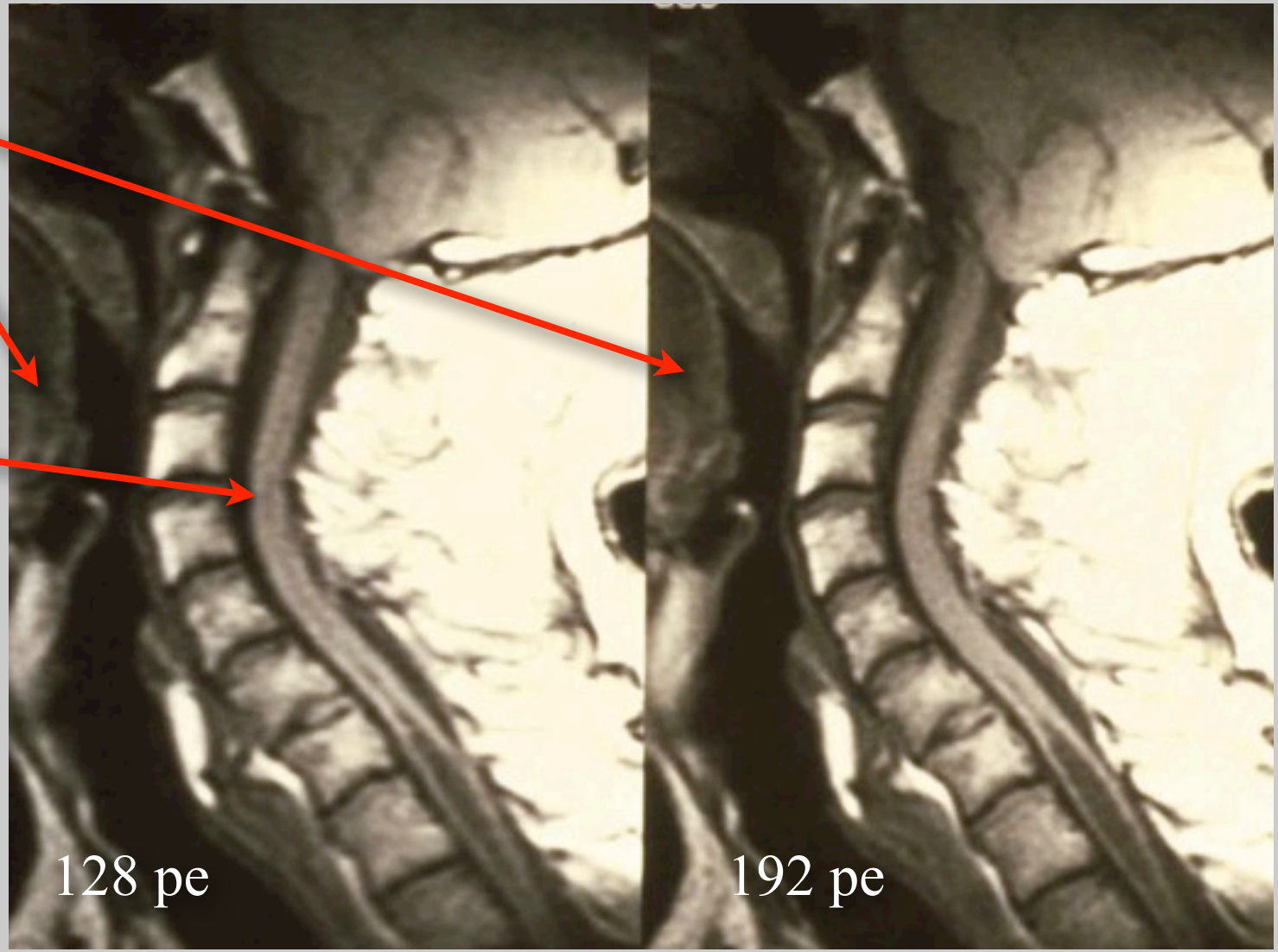
- Side-Lobes cause ripple artifacts (Gibbs)
- Width of main lobe determines the resolution
  - Common measure is full-width at half max
  - For simplicity we use zero-crossing of sinc

$$x = \frac{n}{W_{kx}} \quad \Rightarrow \quad \delta_x = \frac{1}{W_{kx}}$$

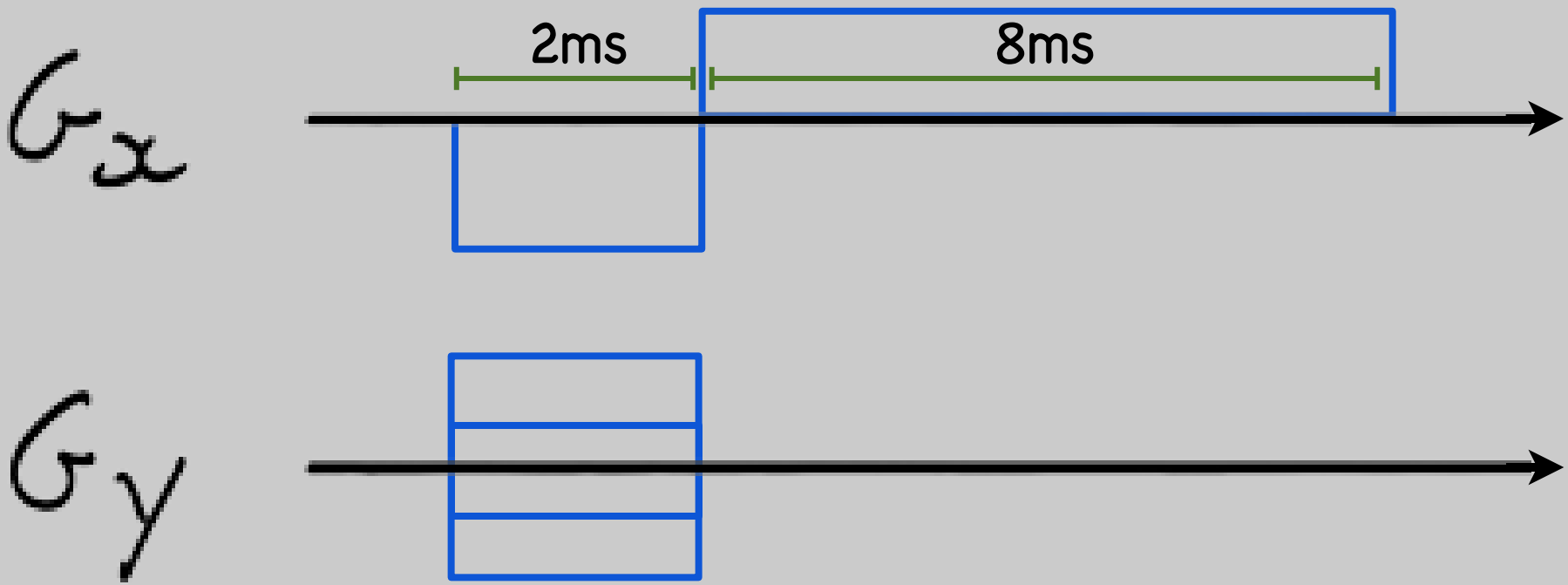
Aliasing and Gibbs Ringing

Aliasing

Gibbs ringing appears as Syrinx artifact



Example: 2DFT Pulse Sequence Design



- Q: What are the amplitudes for $G_x,\ G_{x,pr},\ G_y$
  $FOV_x = FOV_y = 25.6$ cm
  $\delta_x = \delta_x = 0.1$ cm

Solution:

First:

$$N_p = N_r = \frac{25.6}{0.1} = 256 \quad \text{\#of samples}$$

Next:

$$W_{kx} = \frac{1}{\delta x} = \frac{1}{0.1} = 10 \ \text{cm}^{-1}$$

$$(\pm 5)$$

Then:

$$\frac{\gamma}{2\pi} G_x \tau_x = 10$$

$$G_x = \frac{10}{(4.257 \ \ \text{KHz/G})(8\text{ms})} \approx 0.3 \ G/cm$$

Solution

The prewinder has the same area as half the readout:

$$
\begin{aligned}
(G_{x,pre}) \cdot (2 \text{ ms}) &= (G_x)\left(\frac{8 \text{ ms}}{2}\right) \\
G_{x,pre} &= (0.3 \text{ G/cm})\frac{4 \text{ ms}}{2 \text{ ms}} \approx 0.6 \text{ G/cm}
\end{aligned}
$$

Solution

In the Y dimension:

$$\delta_y = 0.1 \text{ cm}$$

so,

$$W_{ky} = \frac{1}{\delta_y} = 10 \text{ cm}^{-1}$$

Phase-encode gradients integrate to $\pm W_{ky}/2$

$$\frac{\gamma}{2\pi} G_y \tau_y = 5 \text{ cm}^{-1}$$

$$G_y \approx 0.6 \text{ G/cm}$$

Parameters

SIMPLER TO REMEMBER

EXTENT IN ONE DOMAIN (FOV, W_k)

= INVERSE ~~OF~~ IN OTHER DOMAIN (δ, Δk)

RESOLUTION $\quad$ 1/Extent

$$\Delta k = \frac{1}{FOV}$$

$$\delta = \frac{1}{W_k}$$

EXAMPLE

10 CM FOV $\Rightarrow$ $\Delta k = 0.1 \ \frac{\text{cycles}}{\text{cm}}$

0.1 CM RES. $\Rightarrow$ $W_k = 10 \ \frac{\text{cycles}}{\text{cm}}$.

A few Practicalities

- DFT Reconstruction
- Magnitude Reconstruction
- Truncation artifacts
- Square Pixels

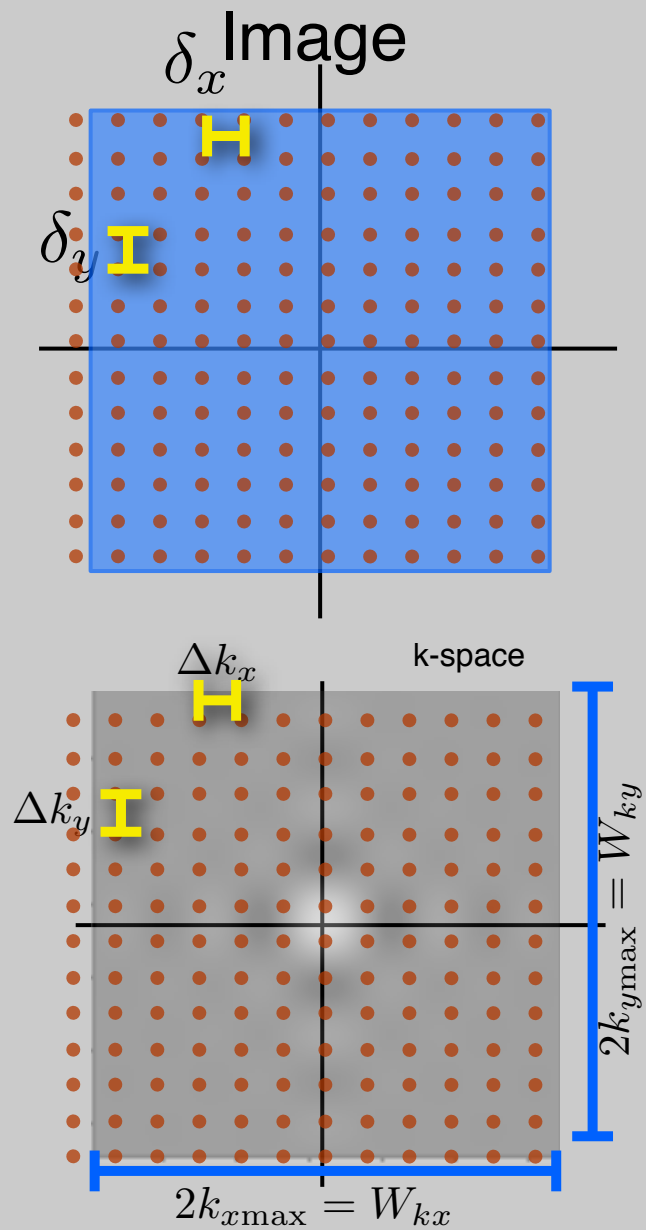
DFT Reconstruction

- Data is on a 2D Grid in k-space
- Image is on a 2D grid too

$$\tilde{m}(u,v) = \hat{m}(u\Delta k_x, v\Delta k_y)$$
$$\tilde{M}(a,b) = \hat{M}(a\delta x, b\delta y)$$

$$a,u \quad \in \quad \left\{ -\frac{N_r}{2} : \frac{N_r}{2} - 1 \right\}$$

$$b,v \quad \in \quad \left\{ -\frac{N_p}{2} : \frac{N_p}{2} - 1 \right\}$$



DFT Reconstruction

$$\tilde{m}(u,v) = \hat{m}(u\Delta k_x, v\Delta k_y) \qquad a,u \in \left\{-\frac{N_r}{2} : \frac{N_r}{2} - 1\right\}$$

$$\tilde{M}(a,b) = \hat{M}(a\delta x, b\delta y) \qquad b,v \in \left\{-\frac{N_p}{2} : \frac{N_p}{2} - 1\right\}$$

$$\tilde{M}(a,b) = \sum_u \sum_v \tilde{m}(u,v) e^{+i2\pi\left(\frac{ua}{N_r} + \frac{vb}{N_p}\right)}$$

See derivation in text

FFT Implementations start at:

$$a,u \in \{0 : N_r - 1\}$$

$$b,v \in \{0 : N_p - 1\}$$

DFT Reconstruction

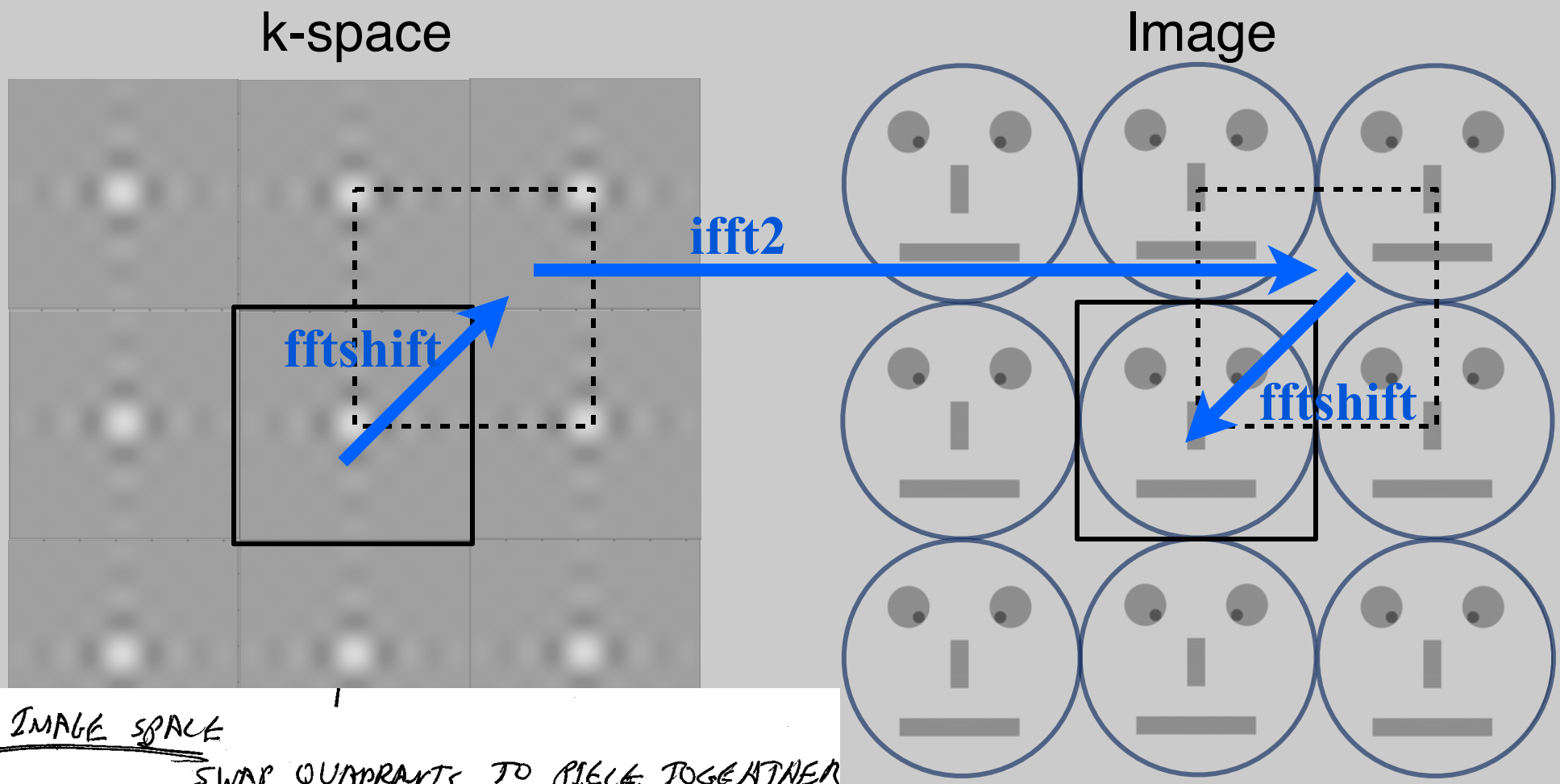


IMAGE SPACE

SWAP QUADRANTS TO PIECE TOGETHER

IN MATLAB:

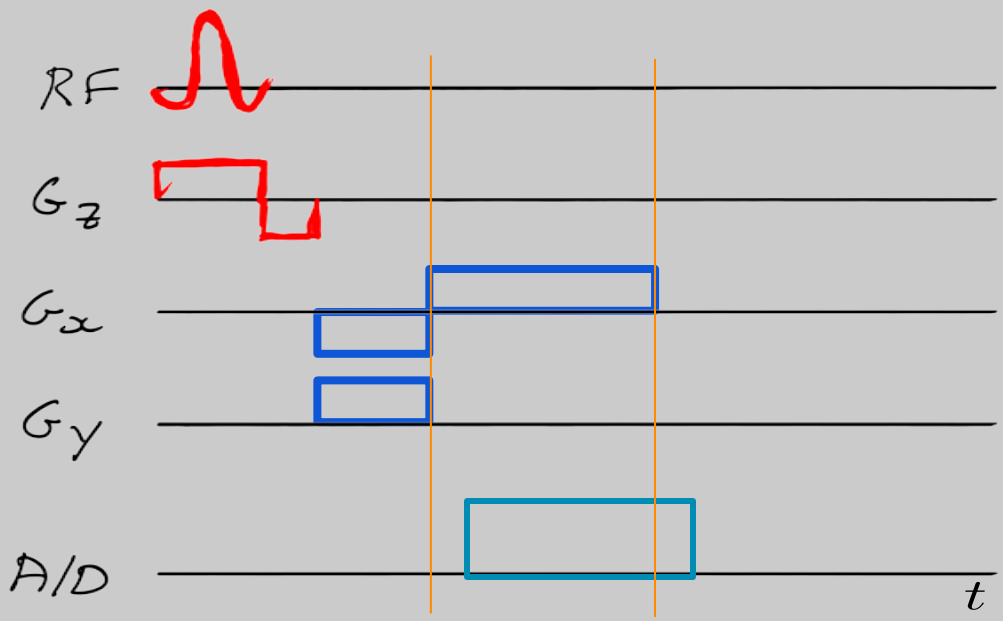
FFTSHIFT(IFFT2(IFFTSHIFT($\hat{m}$)))

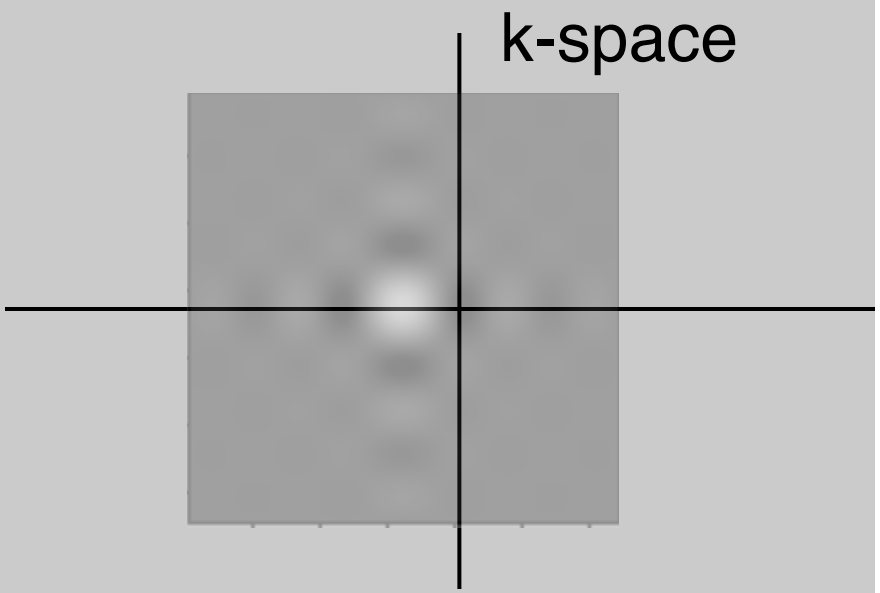
A few Practicalities

- DFT Reconstruction
- Magnitude Reconstruction
- Truncation artifacts
- Square Pixels

Magnitude Reconstruction

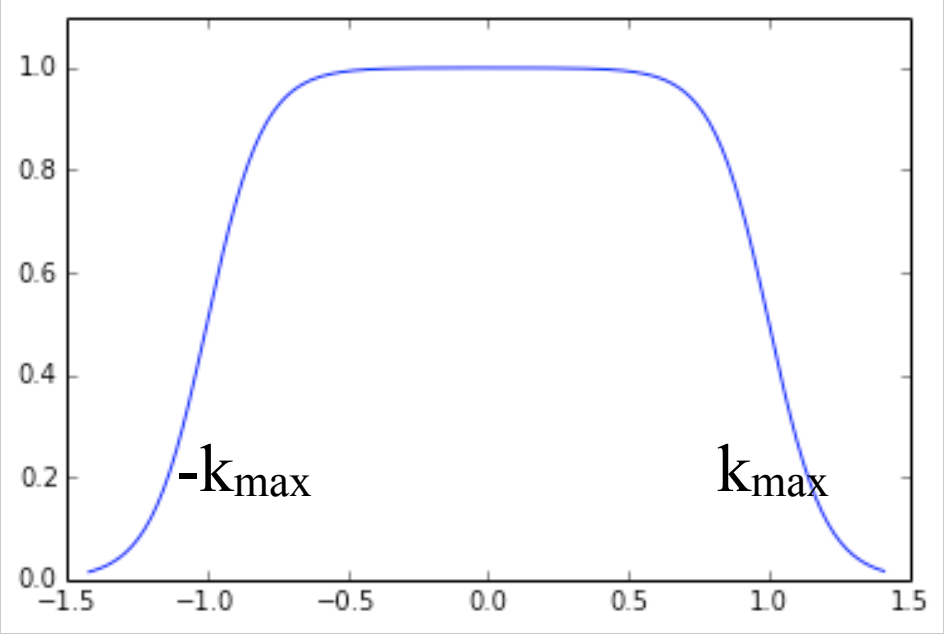
- Most MR Images are Magnitude
- In many cases, images should be real, but:
  - Timing Errors
  - Frequency Variations
  - Coil Sensitivities

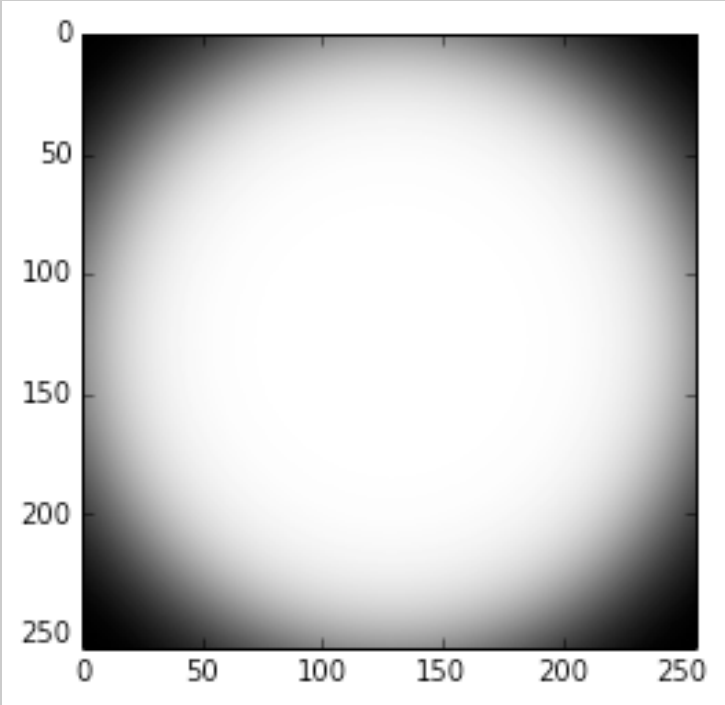




Truncation Artifacts

- Apodization
  - K-space is usually multiplied by a circular 2D window





  - Resolution independent of angle
  - reduced Gibbs ringing

Square Pixels

- MR Images are displayed in a matrix in which pixels have same x-y size
- Data does not often meets it
- Example: N_p=128, N_f=256
- W_{ky} = 10 cm^{-1}, W_{kx} = 5 cm^{-1}
- Same FOV_x=FOV_y, resolution different