Principles of MRI

EE225E / BIO265

Chapter 05
MRI Haiku

Recon runs all night
k-space, image space now done
oh oh, artifacts!

D. Nishimura 2012
The Bloch Equation

- Combine Precession and relaxation

\[
\frac{d\vec{M}}{dt} = -\gamma \vec{B} \times \vec{M} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{M_z - M_0}{T_1} \hat{k}
\]

- Phenomenological: Fits observations
  - Describes most of MRI
  - Sometimes Fails.... J-coupling, Magn, transfer
Mathematical Description of MRI

• Plan:

1) Derive Math for each element
2) Put together: e.g., the BLOCH equation
3) Solve the Bloch eqn. for special cases
   a) Excitation         CH. 6 (later)
   b) Reception          CH. 5 (first)
      i) Derive k-space (AGAIN!!!)
      ii) Pulse sequence
      iii) Sampling
The Bloch Equation

- In the rotating frame: \( \vec{\rho} = [x, y, z]^T \)

\[
\vec{B} = \gamma \vec{G} \cdot \vec{\rho} \hat{k} + B_{1x} \hat{i} + B_{1y} \hat{j}
\]

\[
\frac{d\vec{M}}{dt} = -\gamma \vec{B} \times \vec{M} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{M_z - M_0}{T_1} \hat{k}
\]

- Precession
- Transverse decay
- Longitudinal recovery
The Bloch Equation

• In Matrix Form:

\[
\begin{bmatrix}
\dot{M}_x \\
\dot{M}_y \\
\dot{M}_z
\end{bmatrix}
= \begin{bmatrix}
0 & \gamma \vec{G} \cdot \vec{r} & -\gamma B_{1y} \\
-\gamma \vec{G} \cdot \vec{r} & 0 & \gamma B_{1x} \\
\gamma B_{1y} & -\gamma B_{1x} & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
+ \begin{bmatrix}
\frac{1}{T_2} & 0 & 0 \\
0 & -\frac{1}{T_2} & 0 \\
0 & 0 & -\frac{1}{T_1}
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\frac{1}{T_1}
\end{bmatrix}
\begin{bmatrix}
M_0
\end{bmatrix}
\]
Bloch Equation

- Combined (rotating frame)

\[
\begin{bmatrix}
\dot{M}_x \\
\dot{M}_y \\
\dot{M}_z
\end{bmatrix}
= \begin{bmatrix}
-\frac{1}{T_2} & \gamma \vec{G} \cdot \vec{r} & -\gamma B_{1y} \\
-\gamma \vec{G} \cdot \vec{r} & -\frac{1}{T_2} & \gamma B_{1x} \\
\gamma B_{1y} & -\gamma B_{1x} & -\frac{1}{T_1}
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\frac{1}{T_1}
\end{bmatrix} M_0
\]

- T1 is the source of all signals!
- Magnetization distribution unknown
- Can probe by changing B1 and G

Rotation + relaxation

Recovery
Solving the Bloch Equation

Two Special Cases:

• Reception
  – Data acquisition
  – Spatial encoding
  – Explicit solutions! (Today)

• Excitation
  – Non linear problems
  – No general solution
  – Many solutions for special cases (Ch. 6)

\[ B_{1x} = 0, \ B_{1y} = 0 \]

\[ B_{1x} \neq 0, \ B_{1y} \neq 0 \]
Reception

- Magnetization is a function of $\vec{r}$ and $t$. Also, assume single, constant $T_1$ and $T_2$.

$$\vec{M}(\vec{r}, t) = [M_x(\vec{r}, t), M_y(\vec{r}, t), M_z(\vec{r}, t)]^T$$

- We would like to resolve this using a time varying gradient

$$\vec{G}(t) = [G_x(t), G_y(t), G_z(t)]^T$$
• With no \( B_1(t) \):

\[
\begin{bmatrix}
\dot{M}_x \\
\dot{M}_y \\
\dot{M}_z 
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{T_2} & \gamma \vec{G} \cdot \vec{r} & 0 \\
-\gamma \vec{G} \cdot \vec{r} & -\frac{1}{T_2} & 0 \\
0 & 0 & -\frac{1}{T_1}
\end{bmatrix} \begin{bmatrix}
M_x \\
M_y \\
M_z 
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\frac{1}{T_1}
\end{bmatrix} M_0
\]
• With no B1(t):

\[
\begin{bmatrix}
\dot{M}_x \\
\dot{M}_y \\
\dot{M}_z
\end{bmatrix} =
\begin{bmatrix}
-\frac{1}{T_2} & \gamma \tilde{G}(t) \cdot \vec{r} & 0 \\
-\gamma \tilde{G}(t) \cdot \vec{r} & -\frac{1}{T_2} & 0 \\
0 & 0 & -\frac{1}{T_1}
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\frac{1}{T_1}
\end{bmatrix} M_0
\]

• Decouples into two independent eqns.

1. \[
\begin{bmatrix}
\dot{M}_x \\
\dot{M}_y
\end{bmatrix} =
\begin{bmatrix}
-\frac{1}{T_2} & \gamma \tilde{G}(t) \cdot \vec{r} \\
-\gamma \tilde{G}(t) \cdot \vec{r} & -\frac{1}{T_2}
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y
\end{bmatrix}
\]

2. \[
\dot{M}_z = -\frac{1}{T_1} M_2 + \frac{1}{T_1} M_0
\]
Solution for Reception

• Mz: (2) is already solved!

\[ M_z(\vec{r}, t) = M_0(\vec{r}) + \left[ M_z(\vec{r}, 0) - M_0(\vec{r}) \right] e^{-\frac{t}{\tau}} \]

\[ = M_z(\vec{r}, 0) e^{-\frac{t}{\tau}} + (1 - e^{-\frac{t}{\tau}}) M_0(\vec{r}) \]

Initial Magn. \ \underline{Decays} \ \ \ EQUILIBRIUM MAGN. \ \underline{RESTORED}
Solution for Reception

• Mxy: (1)

\[
\begin{pmatrix}
\dot{M}_x \\
\dot{M}_y
\end{pmatrix} = \begin{pmatrix}
-\frac{i}{\Delta} & \vec{SG}(t) \cdot \vec{r} \\
-\vec{SG}(t) \cdot \vec{r} & -\frac{i}{\Delta}
\end{pmatrix} \begin{pmatrix}
M_x \\
M_y
\end{pmatrix}
\]

Take eqn (\(\circ\)) + i eqn (\(\square\))

\[\dot{M}_x + i \dot{M}_y = -\frac{i}{\Delta} M_x + \vec{SG}(t) \cdot \vec{r} M_y - i \vec{SG}(t) \cdot \vec{r} M_x - \frac{i}{\Delta} M_y = \]

\[= -\frac{i}{\Delta} (M_x + i M_y) + i \vec{SG}(t) \cdot \vec{r} i M_y - i \vec{SG}(t) \cdot \vec{r} M_x + i s - i t^a \]
Solution for Reception

\[ \dot{M}_{xy} = \left( -\frac{1}{T_2} - i\gamma \vec{G}(t) \cdot \vec{r} \right) M_{xy} \]

\[ \Rightarrow 0 = \dot{M}_{xy} + \left( \frac{1}{T_2} + i\gamma \vec{G}(t) \cdot \vec{r} \right) M_{xy} \]

- Can be solved using an integrating factor
Differential Equations 101

• Given an ODE

\[ y' + p(x)y = Q(x) \]

• Multiply with integrating factor \( M(x) \)

\[ M(x)y' + M(x)p(x)y = M(x)Q(x) \]

• So,

\[ (M(x)y)' = M(x)Q(x) \]

• And,

\[ y(x)M(x) = \int Q(x)M(x) + C \]
Solution for Reception

\[
\dot{M}_{xy} = \left( -\frac{1}{T_2} - i\gamma \vec{G}(t) \cdot \vec{r} \right) M_{xy}
\]

\[
\Rightarrow 0 = \dot{M}_{xy} + \left( \frac{1}{T_2} + i\gamma \vec{G}(t) \cdot \vec{r} \right) M_{xy}
\]

• Can be solved using an integrating factor

\[
e^{\frac{t}{T_2} + i\gamma \vec{r} \cdot \int_0^t \vec{G}(\tau) d\tau}
\]
Solution for Reception

\[ \dot{M}_{xy} + \left( \frac{1}{T_2} + i\gamma \vec{G}(t) \cdot \vec{r} \right) M_{xy} = 0 \]

\[ \dot{M}_{xy} e^{\frac{t}{T_2}} + i\gamma \vec{r} \cdot \int_0^T \vec{G}(\tau) d\tau + \left( \frac{1}{T_2} + i\gamma \vec{G}(t) \cdot \vec{r} \right) M_{xy} e^{\frac{t}{T_2}} + i\gamma \vec{r} \cdot \int_0^T \vec{G}(\tau) d\tau = 0 \]

\[ \left( M_{xy} e^{\frac{t}{T_2}} + i\gamma \vec{r} \cdot \int_0^t \vec{G}(\tau) d\tau \right)' = 0 \]

• Integrate:

\[ M_{xy}(\vec{r}, t)e^{\frac{t}{T_2}} + i\gamma \vec{r} \cdot \int_0^t \vec{G}(\tau) d\tau = M_{xy}(\vec{r}, 0) \]
Solution for Reception

\[ M_{xy}(\vec{r}, t) = M_{xy}(\vec{r}, 0) e^{-\frac{t}{T_2}} e^{-i\vec{r} \cdot \mathbf{G}(\tau) d\tau} \]

Define a Spatial frequency vector:

\[ \vec{k}(t) = [k_x(t), k_y(t), k_z(t)]^T \]

\[ \vec{k}(t) = \frac{\gamma}{2\pi} \int_0^t \mathbf{G}(\tau) d\tau \]
Solution for Reception

- Transverse Magnetization $M_{xy}$ is then:

$$M_{xy}(\vec{r}, t) = M_{xy}(\vec{r}, 0)e^{-\frac{t}{T_2}}e^{-i2\pi \vec{k}(t) \cdot \vec{r}}$$

- Received Signal is proportional to $M_{xy}$ integrated over volume:

$$s(t) = \int_{\bar{R}} M_{xy}(\vec{r}, 0)e^{-\frac{t}{T_2}}e^{-i2\pi \vec{k}(t) \cdot \vec{r}} d\vec{r}$$

<table>
<thead>
<tr>
<th>Fourier kernel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Also a function of $r$!</td>
</tr>
</tbody>
</table>
Signal Equation

• Assume that T2 is LARGE

\[ s(t) \approx \int_{\mathbb{R}} M_{xy}(\vec{r}, 0) e^{-i2\pi \vec{k}(t) \cdot \vec{r}} d\vec{r} \]

• So,

\[ s(t) = \left. \mathcal{F} \left\{ M_{xy}(\vec{r}, 0) \right\} \right|_{\vec{k} = \vec{k}(t)} \]

samples of FT of \( M_{xy}(\vec{r}, 0) \)
along a trajectory
k-Space

\[ s(t) = \mathcal{F} \left\{ M_{xy}(\vec{r}, 0) \right\} \bigg|_{\vec{k} = \vec{k}(t)} \]

samples of FT of \( M_{xy}(\vec{r}, 0) \) along a trajectory

\[ \vec{k}(t) \text{ is } k\text{-space trajectory} \]

\[ \vec{K}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{g}(\gamma) \, d\gamma \]

\( \ast \) area of gross sent waveform
k-Space Trajectories

- We control $G(t) \Rightarrow$ control $\dot{R}(t)$
- Amplitude limited
  - $\sim 4 \, \text{cm}$ on whole body sys. ($17 \, \text{kHz} / \text{cm}$)
  - $k'(t)$ limited
  - Limited k-space “velocity”
- $\frac{d}{dt} \dot{G}(t)$ limited (slew-rate)
k-Space Trajectories

- Limited rate of change of $\hat{\mathbf{r}}(t)$
  - Called slew-rate
  - $\sim 15 \text{ (cm/s)} / \text{ms}$ or $150 \text{ Tm/s}$ (SR150)
  - $\sim 267 \mu s$ rise-time
- $k''(t)$ limited $\Rightarrow$ acceleration curvature
  - $\hat{\mathbf{r}}(t)$ will be smooth, continuous
Which trajectory should we choose?
Slice Selective 2D Example

- Slice Select in Z
- Resolve in x,y
- Need $G_x(t), G_y(t)$
- Acquire $k_x(t), k_y(t)$

$$s(t) = \iiint_{x,y,z} M_{xy}(\vec{r},0) e^{-i \vec{k} \cdot \vec{r}(t)} \, d\vec{r}$$

$$k_z(t) = 0 \Rightarrow \iiint_{x,y,z} \left[ \int_{z} M_{xy}(\vec{r},0) \, dz \right] e^{-i \vec{k} \cdot \vec{r}(t)} \, d\vec{r}$$
k-Space Trajectory

- Extent $\Rightarrow$ Resolution
- Density $\Rightarrow$ FOV (next time)
- Often Square coverage
  - Sometimes circle
- What are some options?

M. Lustig, EECS UC Berkeley
Radial Scanning (Projection Reconstruction)

- **RF**
- **$G_z$**
- **$G_x$**
- **$G_y$**
- **$k_x$**
- **$A/D$**

- **Image**
- **k-space**

**Questions:**
- What is the k-space trajectory?
- What is $s(t)$ A/D output?
Radial Scanning (Projection Reconstruction)

what is $s(t)$ A/D output?
Radial Scanning (Projection Reconstruction)

RF

$G_z$

$G_x$

$G_y$

$k_x$

A/D

→optional

Image

k-space
Radial Scanning (Projection Reconstruction)

- Repeat for each diameter
- Peak at $k(t) = 0$
2DFT (Spin Warp)

RF

$G_z$

$G_x$

$G_y$

A/D

$k_{x}(t)$

$k_{y}(t)$
2DFT (Spin Warp)

RF

G\(_{z}\)

G\(_{x}\)

G\(_{y}\)

A/D

readout/freq.-encode

pre-winder

phase-encode

echo-time (TE)

k\(_{x}(t)\)

k\(_{y}(t)\)

k-space

M. Lustig, EECS UC Berkeley
2DFT (Spin Warp)

- Repeat for each line
- Obtain 2D Cartesian Grid
- By far the most common trajectory!
2DFT (Spin Warp)

- Repeat for each line
- Obtain 2D Cartesian Grid
- By far the most common trajectory!

Overlap prewinders with slice refocus
2DFT (Spin Warp)

- Gy gradient is **Phase-Encode** gradient
  - Establishes fixed linear-phase before readout

\[ M_{xy}(r,0) e^{-i\omega \pi ky(t_1)y} \]

- Gx gradient is **Frequency-Encode** gradient, or Readout

\[ M_{xy}(r,0) e^{-i\omega \pi ky(t_1)y} e^{-i\omega \pi k_x(t_1)x} \]

M. Lustig, EECS UC Berkeley
What Does This Sequence Do?

RF

\[ k_x(t) \]

\[ k_y(t) \]
Multi-Echo 2DFT

- Echo-Time is when $k_x(t) = 0$

$RF$  
$G_2$  
$G_x$  
$G_y$  
$A/D$

$TE_1$  $TE_2$  $TE_3$

$k_x(t)$  $k_y(t)$

$TE_1$  $TE_2$  $TE_3$
Echo Planar Imaging (EPI)

- More phase encode for each excitation

\[
\begin{align*}
RF & \quad G_z & \quad G_x & \quad G_y \\
& & & \\
& t & t & t \\
& k_x(t) & k_y(t) & \\
& t & t & t
\end{align*}
\]
Echo Planar Imaging (EPI)

- More phase encode for each excitation

\[ k_x(t) \]
\[ k_y(t) \]
Spiral

$R_F$

$G_z$

$G_x$

$G_y$

A/D

$k_x(t)$

$k_y(t)$
EPI and Spiral

- **EPI**: Most common high-speed acquisition.
  - fMRI, diffusion, real-time
  - Can use interleaved multiple shots
  - “Easy” DFT reconstruction

- **Spiral**
  - Very hardware efficient
  - Useful for flow/hearts, also used for fMRI
  - Often multiple interleaved shots
  - Uses gridding for non-uniform FT reconstruction
Sampling, Resolution and FOV

• Signal is sample of FT of object

\[ s(t) = \widetilde{\mathcal{F}} \left\{ M_{xy}(\mathbf{r},0) \right\} \bigg|_{\mathbf{k} = \mathbf{k}(t)} \]

• sampled at discrete times!

\[ s(n \Delta t) = m_{xy}(\mathbf{k}(n \Delta t)) \]
2DFT Case

- $\Delta k_x$ and $\Delta k_y$

- $2k_{x\text{max}} = W_{kx}$

- $2k_{y\text{max}} = W_{ky}$

- $G_x$ and $G_y$

- $\tau_1$ and $\tau_2$

- Nr - # readout samples
- Np - # phase encodes

- $k_{y\text{max}} = \frac{\gamma}{2\pi} G_y \tau_1$
- $W_{kx} = \frac{\gamma}{2\pi} G_x \tau_2$
- $\Delta k_x = \frac{W_{kx}}{N_r}$
- $\Delta k_y = \frac{W_{ky}}{N_p}$
Sampled Fourier Transform

- In k-space

\[ \hat{m}_{xy}(k_x, k_y) = m(k_x, k_y) \left[ \frac{1}{\Delta k_x \Delta k_y} \mathcal{F}\left( \frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y} \right) \right] \cdot \tilde{\mathcal{M}} \left( \frac{k_x}{W_{k_x}}, \frac{k_y}{W_{k_y}} \right) \]

  - F.T of object
  - Sampling
  - Extent

- In Image space

\[ \hat{M}_{xy}(x, y) = M_{xy}(x, y) * * \mathcal{M}\left( \Delta k_x x, \Delta k_y y \right) * * W_{k_x} W_{k_y} \text{sinc}(W_{k_x} x) \text{sinc}(W_{k_y} y) \]

  - Cont. IMAGE
  - Replication/aliasing
  - Blurring/Resolution

M. Lustig, EECS UC Berkeley
FOV and Aliasing

• Assume infinite resolution (for now...)

\[ W_{kx} \cdot W_{ky} \cdot \text{Sinc}(W_{kx}x) \cdot \text{Sinc}(W_{ky}y) = \delta(x, y) \]

then,

\[ \hat{M}_{xy}(x, y) = M_{xy}(x, y) \ast \ast \tilde{W}(\Delta k_x, \Delta k_y) \]

• \( \tilde{W}(\Delta k_x, \Delta k_y) \) has an impulse when:

\[ \Delta k_x = n \quad \Rightarrow \quad x = \frac{n}{\Delta k_x} \quad m,n \text{ integers} \]

\[ \Delta k_y = m \quad \Rightarrow \quad y = \frac{m}{\Delta k_y} \quad m,n \text{ integers} \]
FOV and Aliasing

\[ \Delta k_x = n \Rightarrow x = \frac{n}{\Delta k_x} \]
\[ \Delta k_y = m \Rightarrow y = \frac{m}{\Delta k_y} \]

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]
\[ \text{FOV}_y = \frac{1}{\Delta k_y} \]

no aliasing as long as object is smaller than FOV
FOV and Aliasing

\[ \Delta k_x = n \Rightarrow x = \frac{n}{\Delta k_x} \]
\[ \Delta k_y = m \Rightarrow y = \frac{m}{\Delta k_y} \]

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]
\[ \text{FOV}_y = \frac{1}{\Delta k_y} \]

no aliasing as long as object is smaller than FOV
FOV and Aliasing

$\Delta k_x = n \Rightarrow x = \frac{n}{\Delta k_x}$

$\Delta k_y = m \Rightarrow y = \frac{m}{\Delta k_y}$

$FOV_x = \frac{1}{\Delta k_x}$

$FOV_y = \frac{1}{\Delta k_y}$

no aliasing as long as object is smaller than FOV
Aliasing in 2DFT

- Readout is sampling a continuous time signal
- Phase encode is inherently discrete!
- Aliasing does not occur in frequency encode direction
In practice: No Aliasing Along Readout

\[ s(t) \rightarrow \text{LPF} \rightarrow x \rightarrow s(n\Delta t) \]

\[ \mathcal{W}(t/\Delta t) \]

- Anti-aliasing
- No aliasing

\[ f [Hz], x [cm] \]
FOV

• In readout/frequency-encode direction
  – Increase FOV by increasing sampling rate
  – No change in acquisition gradients
  – Just more data
  – FOVx is free (almost) → no aliasing

• In phase encode, aliasing can occur
  – Phase-encodes are fundamentally discrete
  – $\text{FOV}_y \uparrow \Rightarrow \Delta k_y = \frac{W_{ky}}{N_p} \Rightarrow N_p \uparrow$
  – More acquisitions → more scan time
  – $\text{FOV}_y$ costs scan time!
Q: What would the reconstruction image look like?
Q: What is the difference in acquisition between the two images?
Figure 36: Breathhold postgadolinium MRI in a 1 year old male with hypertension using oX acceleration at 1.2mm resolution. Left images cag cd are with fLRC and right cbg dd are with LlhSPIRiT compressed sensing. Note improved delineation of pancreas cbig arrowd, pancreatic duct cmiddle arrowd, and diaphragm csmall arrowd. Left gastric artery emerges from the noise.

6.3x, ARC 70 seconds 6.3x, SPIRiT 70 seconds

Figure 37: nD SPGR g a r minute cartilage sequence too lengthy for routine use now with qin fold acceleration. Note restored delineation of growth plate carrowheadd and a nonossifying fibroma with SPIRiTi.

Figure 38: Representative images from nD Tm weighted scan at 1.1mm resolution with Poissonhdisc sampling and phfold outer acceleration in an s day old female with left isomerism and absent portal vesi. fLRC images are too noisy. Images reconstructed with SPIRIT show decreased noise and improved structure delineation. zoomed insets in cad andc bd show mesenteric veins carrowd and liver capsule carrowheadd. zoomed insets in ccd and cdd show left gastric artery cthick arrowd and branch hepatic artery cdashed arrowdi. In ced and cfd show aliased peripheral IV tubing ccaratdg with true position shown in the localizer cgdi.

ai equivalent or improved SNR bi less motion artifactg and ci equivalent or improved delineation of specific anatomic structures over standard methodsi.

Table 2: MRI protocol to validate technih with parametersu flip angle lp degreesg FOV nk cmg matrix nmk x mmog slice thickness p mmg qk slicesg scan time approxh approximately np secondsi.

tm sequences with parametersu FOV nk cmg matrix nmk x mmog slice thickness n mmg qk slicesg scan time approxp minutesi.

Subjects: mp consecutive patients referred for contrastenhanced abdominal MRI will be recruited i.

Design: Each patient will undergo an MRI protocolg as shown in Tabi mi. Comparison of conventional techniques with the experimenh DimilhDimin will be performedg as shown in Figi nti. For each sequence the central calibration portion of kh space will be acquired twice in interleaved fashiong iiei each phase encode cfor Tm imagingd or echo train cTm imagingd will be repeated twice backhtohbacki. The khspace data can then be subsampled by a factor of two in a Cartesian fashion in two waysg yielding two disjoint datasetsi. The two cartesian images can then be assessed for SNR using the difference method [rk]i. Similarlyg two disjoint Poissonh khspace data sets can be createdg and reconstructed with moh.
Aliasing in Radial Trajectory

- Nyquist criteria still determined by FOV
- Cost $\pi$ times more than Cartesian
- However:
  - Aliasing is incoherent
  - Degrades gracefully with reduction of spokes

$$\Delta k < \frac{1}{\text{FOV}}$$
Aliasing in Radial

- 20cm object, 1mm resolution

Point Spread function:

Reconstruction:

- 256 spokes
- 128 spokes
- 64 spokes
Different Samplings

trajectory

PSF

x-section
Spatial Resolution

- The coverage of k-space is finite

\[ \hat{m}_{xy}(k_x, k_y) = m(k_x, k_y) \frac{1}{\Delta k_x \Delta k_y} \mathcal{L}\left(\frac{k_x}{W_{kx}}, \frac{k_y}{W_{ky}}\right) \]

in k-space:

\[ \hat{M}_{xy}(x, y) = M_{xy}(x, y) \ast \mathcal{L}(\Delta k_x x, \Delta k_y y) \ast W_{kx} W_{ky} \text{sinc}(W_{kx} x) \text{sinc}(W_{ky} y) \]

in image space:
Resolution

- Side-Lobes cause ripple artifacts (Gibbs)
- Width of main lobe determines the resolution
  - Common measure is full-width at half max
  - For simplicity we use zero-crossing of sinc

\[ x = \frac{n}{W_{kx}} \implies \delta_x = \frac{1}{W_{kx}} \]
Aliasing and Gibbs Ringing

Aliasing

Gibbs ringing appears as Syrinx artifact

128 pe

192 pe
Example: 2DFT Pulse Sequence Design

Q: What are the amplitudes for

\[ \text{FOV}_x = \text{FOV}_y = 25.6 \text{ cm} \]

\[ \delta_x = \delta_x = 0.1 \text{ cm} \]
Solution:

First:

\[ N_p = N_r = \frac{25.6}{0.1} = 256 \text{ #of samples} \]

Next:

\[ W_{kx} = \frac{1}{\delta x} = \frac{1}{0.1} = 10 \text{ cm}^{-1} \]

(\pm 5)

Then:

\[ \frac{\gamma}{2\pi} G_x \tau_x = 10 \]

\[ G_x = \frac{10}{(4.257 \text{ KHz/G})(8\text{ms})} \approx 0.3 \text{ G/cm} \]
Solution

The prewinder has the same area as half the readout:

\[
(G_{x,\text{pre}}) \cdot (2 \text{ ms}) = (G_x) \left( \frac{8 \text{ ms}}{2} \right)
\]

\[
G_{x,\text{pre}} = (0.3 \text{ G/cm}) \frac{4 \text{ ms}}{2 \text{ ms}} \approx 0.6 \text{ G/cm}
\]
Solution

In the Y dimension:

\[ \delta_y = 0.1 \text{ cm} \]

so,

\[ W_{ky} = \frac{1}{\delta_y} = 10 \text{ cm}^{-1} \]

Phase-encode gradients integrate to \( \pm W_{ky}/2 \)

\[ \frac{\gamma}{2\pi} G_y \tau_y = 5 \text{ cm}^{-1} \]

\[ G_y \approx 0.6 \text{ G/cm} \]
Parameters

SIMPLER TO REMEMBER

EXTENT IN ONE DOMAIN \((\text{FOV}, W_k)\)

= INVERSE IN OTHER DOMAIN \((\delta, D_k)\)

\[
\frac{1}{\text{Extent}}
\]

\[
\text{RESOLUTION}
\]

\[
D_k = \frac{1}{\text{FOV}}
\]

\[
\delta = \frac{1}{W_k}
\]

EXAMPLE

10 cm FOV \(\Rightarrow D_k = 0.1 \text{ cycles/cm}\)

0.1 cm RES. \(\Rightarrow W_k = 10 \text{ cycles/cm}\).
A few Practicalities

- DFT Reconstruction
- Magnitude Reconstruction
- Truncation artifacts
- Square Pixels
DFT Reconstruction

- Data is on a 2D Grid in k-space
- Image is on a 2D grid too

\[ \tilde{m}(u, v) = \tilde{m}(u\Delta k_x, v\Delta k_y) \]
\[ \tilde{M}(a, b) = \tilde{M}(a\delta x, b\delta y) \]

\[ a, u \in \left\{ -\frac{N_r}{2} : \frac{N_r}{2} - 1 \right\} \]
\[ b, v \in \left\{ -\frac{N_p}{2} : \frac{N_p}{2} - 1 \right\} \]
DFT Reconstruction

\[ \tilde{m}(u, v) = \hat{m}(u\Delta k_x, v\Delta k_y) \quad a, u \in \left\{ -\frac{N_r}{2} : \frac{N_r}{2} - 1 \right\} \]

\[ \tilde{M}(a, b) = \hat{M}(a\delta x, b\delta y) \quad b, v \in \left\{ -\frac{N_p}{2} : \frac{N_p}{2} - 1 \right\} \]

\[
\tilde{M}(a, b) = \sum_u \sum_v \tilde{m}(u, v) e^{i2\pi\left(\frac{ua}{N_r} + \frac{vb}{N_p}\right)}
\]

See derivation in text

FFT Implementations start at:

\[ a, u \in \{0 : N_r - 1\} \]

\[ b, v \in \{0 : N_p - 1\} \]
DFT Reconstruction

k-space

fftshift

fft2

Image

fftshift

In MATLAB:

\[ \text{fftshift(fft2(fftshift(\hat{\mathbf{y}})))} \]
A few Practicalities

- DFT Reconstruction
- Magnitude Reconstruction
- Truncation artifacts
- Square Pixels
Magnitude Reconstruction

• Most MR Images are Magnitude
• In many cases, images should be real, but:
  – Timing Errors
  – Frequency Variations
  – Coil Sensitivities
Truncation Artifacts

• Apodization
  – K-space is usually multiplied by a circular 2D window
  – Resolution independent of angle
  – reduced Gibbs ringing
Square Pixels

- MR Images are displayed in a matrix in which pixels have same x-y size
- Data does not often meets it
  - Example: $N_p=128$, $N_f=256$
  - $W_{k_y} = 10$ cm$^{-1}$, $W_{k_x} = 5$ cm$^{-1}$
  - Same $\text{FOV}_x=\text{FOV}_y$, resolution different

DFT

Interpolation