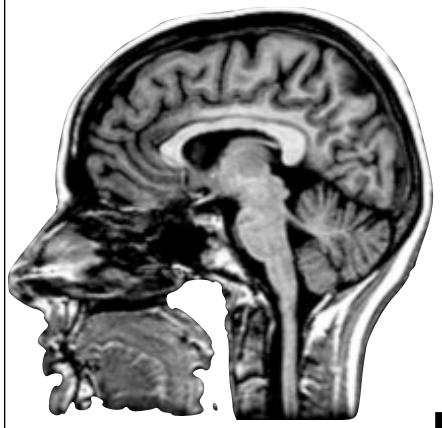
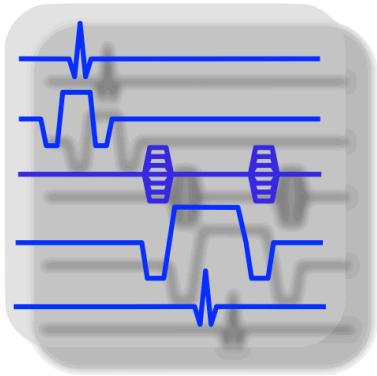




Principles of MRI

EE225E / BIO265

Prerequisites

Instructor: Miki Lustig
UC Berkeley, EECS

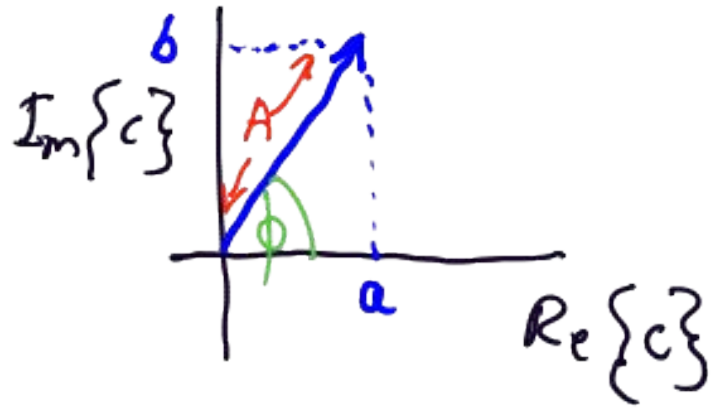
Preliminaries:

complex numbers:

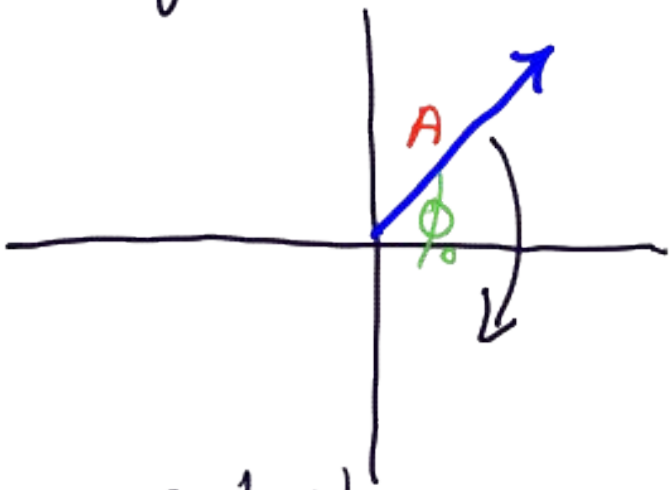
$$c = a + ib \qquad i = \sqrt{-1}$$

or phasors:

$$c = Ae^{i\phi} = A(\cos\phi + i\sin\phi)$$



It is convinient to use complex # to describe a vector in a plane



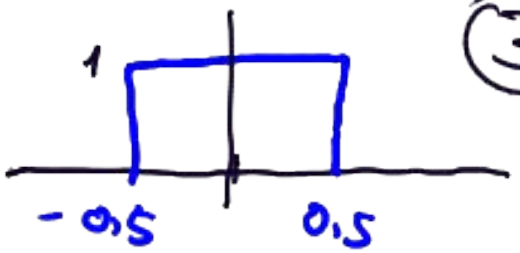
$c(t) = Ae^{-i\omega t + i\phi_0}$ ⇒ clock-wise rotation w/ angular freq. ω

VERY USEFUL TO DESCRIBE

(transverse) MAGNETIZATION

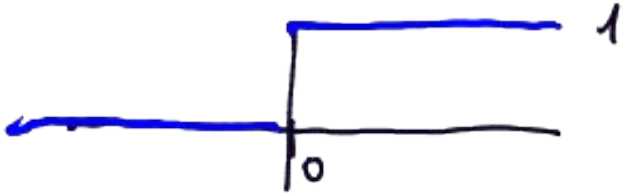
USEFUL FUNCTIONS:

$$\Pi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

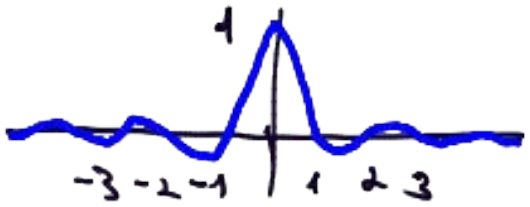


$$\Lambda(x) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



The Impulse function $\delta(x)$

DEFINITION $\Rightarrow$ $\int_{-\infty}^{\infty} \varphi(x)\delta(x)\,dx = \varphi(0)$

Properties:

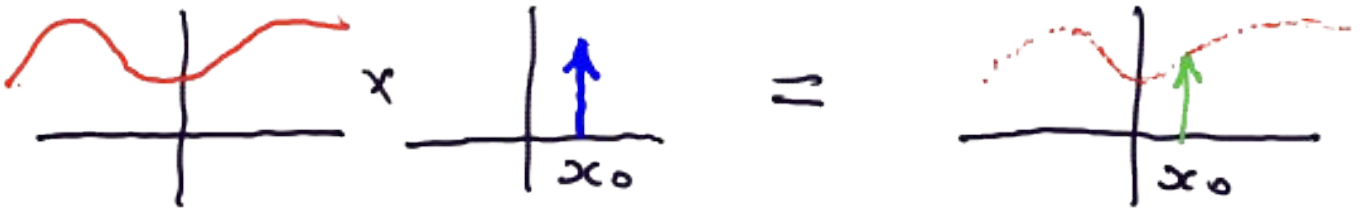
(1) $\delta(x) = 0 \quad \forall x \neq 0$

(2) $\delta(x) \to \infty \quad | \; x = 0$

(3) $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

(4) $\delta(ax) = \frac{1}{|a|}\delta(x)$

(5) $\varphi(x)\delta(x-x_0) = \varphi(x_0)\delta(x-x_0)$

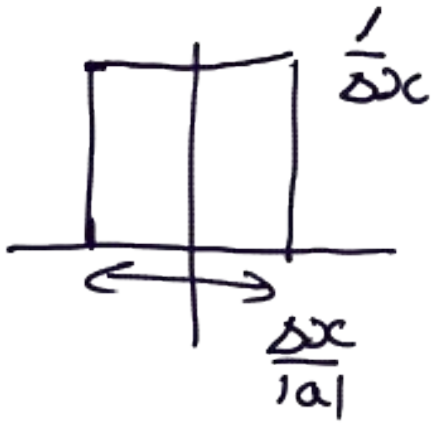


many realizations to $\delta(x)$:

example:

$$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Pi\left(\frac{x}{\Delta x}\right)$$

$$\delta(ax) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Pi\left(\frac{ax}{\Delta x}\right) =$$



$$\text{area} = \frac{\Delta x}{|a|} \cdot \frac{1}{\Delta x} = \frac{1}{|a|}$$

$$= \frac{1}{|a|} \delta(x)$$

Impulse Train $\amalg(x)$

(impulse train
shah function
comb)

$$\amalg(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

what about:

$$\sum_{n=-\infty}^{\infty} \delta(x - \Delta x n) = \sum \delta\left(\Delta x \left(\frac{x}{\Delta x} - n\right)\right) =$$

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$= \frac{1}{\Delta x} \sum \delta\left(\frac{x}{\Delta x} - n\right) = \frac{1}{\Delta x} \amalg\left(\frac{x}{\Delta x}\right)$$

USEFUL TO MODEL SAMPLING!

SYSTEMS

Properties:

Linear

$$T\{g_1(x)+g_2(x)\} = y_1(x)+y_2(x)$$

$$T\{a\,g(x)\} = a\,y(x)$$

shift-invariant

$$T\{g(x-x_0)\} = y(x-x_0)$$

convolution: important math. operator

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(x-\tau)\, h(\tau)\, d\tau$$

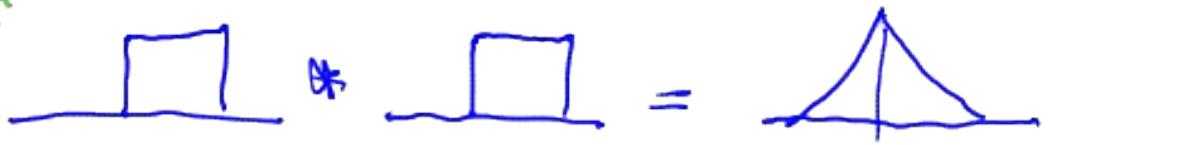
why useful??

$\delta(x) \rightarrow$ [LSI] $\rightarrow h(x)$ Impulse response

$g(x) \rightarrow$ [LSI] $\rightarrow g(x) * h(x)$ for ANY input $g(x)$!

convolution $\Longrightarrow$ blurs

Example



$\Pi(x) * \Pi(x) = \Lambda(x)$

- show example

FOURIER TRANSFORM (1D)

complex val.

$$F(k_x) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi k_x x}\, dx$$

↑ frequency spectrum

HOW COME?

inverse xform:

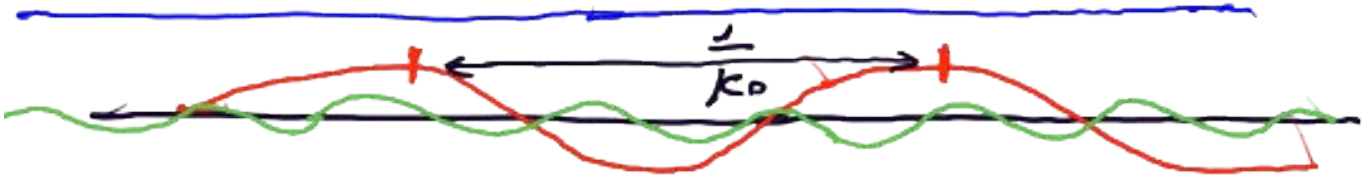
$$f(x) = \int_{-\infty}^{\infty} F(k_x)\, e^{+i2\pi k_x x}\, dk_x$$

Units: If x in cm $\Rightarrow$ k is $1/cm$

t in s $\Rightarrow$ f is Hz

Intuition for FT

$F(k_0)$ IS THE ENERGY OF $f(x)$ at freq. k_0



(-) Amplitude $\Leftrightarrow$ Height

(-) PHASE $\Leftrightarrow$ shift

Some FT

① $\Pi(x) \Rightarrow \mathrm{sinc}(k)$

② $1 \Rightarrow \delta(k)$

$e^{i2\pi k_0 x} \Rightarrow \delta(k-k_0)$

} Periodic ⇒ impulsive

③ $\delta(x-x_0) \Rightarrow e^{i2\pi x_0 k}$ impulsive ⇒ periodic

④ $\amalg(x) \Rightarrow \amalg(k)$

- self transform

impulsive & Periodic ⇒ impulsive & periodic

Properties of Fourier Transform

$f(x) \Rightarrow F(k)$

scaling

1) $f(ax) \Rightarrow \frac{1}{|a|} F\left(\frac{k}{a}\right)$

shrink $\Leftrightarrow$ expand

expand $\Leftrightarrow$ shrink

shift

(2) $f(x-x_0) \Rightarrow e^{-i2\pi x_0 k} F(k)$

shift $\Leftrightarrow$ modulation

modulation

(3) $e^{-i2\pi k_0 x} f(x) \Rightarrow F(k-k_0)$

symmetry

(4) real + even $\Rightarrow$ real + even

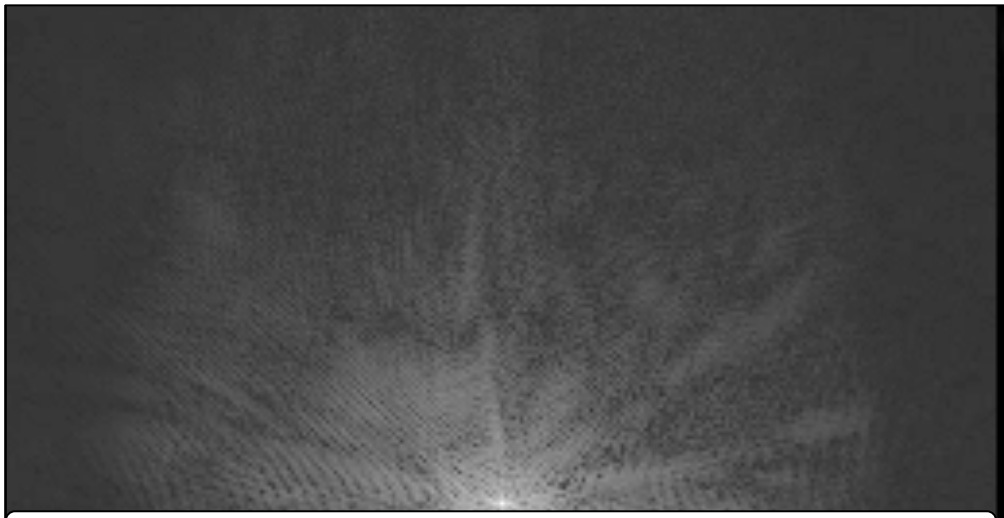
real + odd $\Rightarrow$ imag + odd

$f(x)$ real $\Rightarrow F(-k) = F^*(k)$

what is it good for?

Half Fourier Imaging in MR

k-space (Raw Data)



Image

Complete based on conjugate symmetry
<u>Half the Scan time!</u>

convolution

⑤ $f(x) * g(x) \Rightarrow F(k) \cdot G(k)$

$f(x) \cdot g(x) \Rightarrow F(k) * G(k)$

why useful?

$\mathcal{F}\{f(x) * g(x)\} = F(k)G(k)$

↑ hard ↑ easy

Why Powerful:

often, want specific output so need to design $g(x) \Rightarrow$ "Deconvolution"

easy with FT

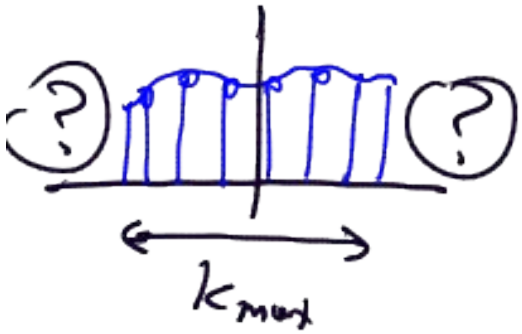
$$Y(k) = G(k)H(k)$$

$$G(k) = \frac{Y(k)}{H(k)}$$

← what is the problem?

EXAMPLE:

suppose $F(k)$ known over limited range



Probelms: ① support

② sampling → later

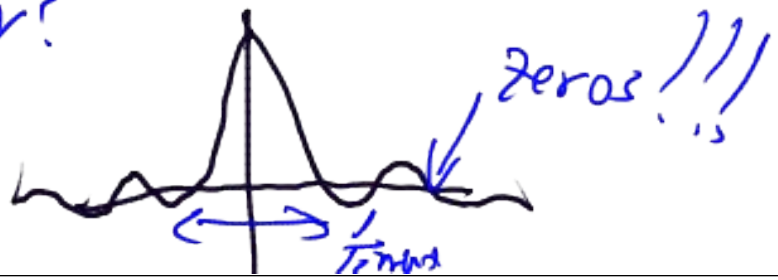
how do we get back $f(x)$?

① we know $F(k)\Pi\left(\frac{k}{k_{max}}\right)$

$\xrightarrow{FT^{-1}}$ $f(x) * k_{max}\,\mathrm{sinc}(k_{max}x)$ blurr!

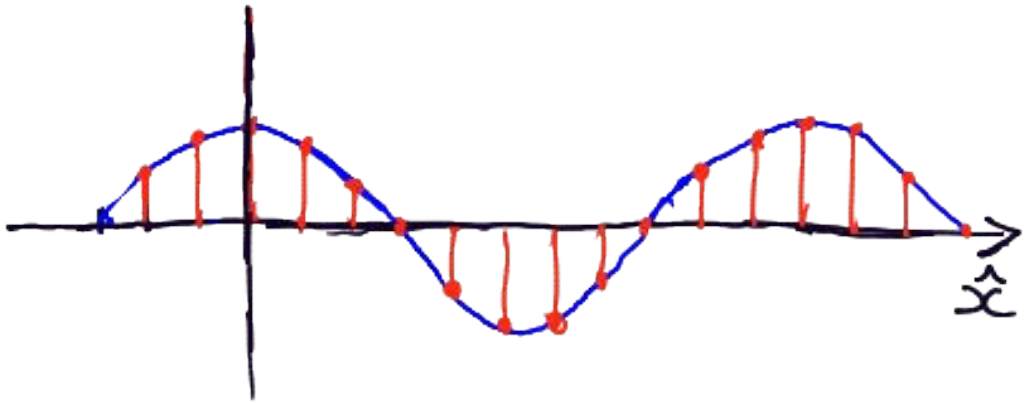
(Q) can you deblurr?

Less blurr $k_{max}\uparrow$



sampling:

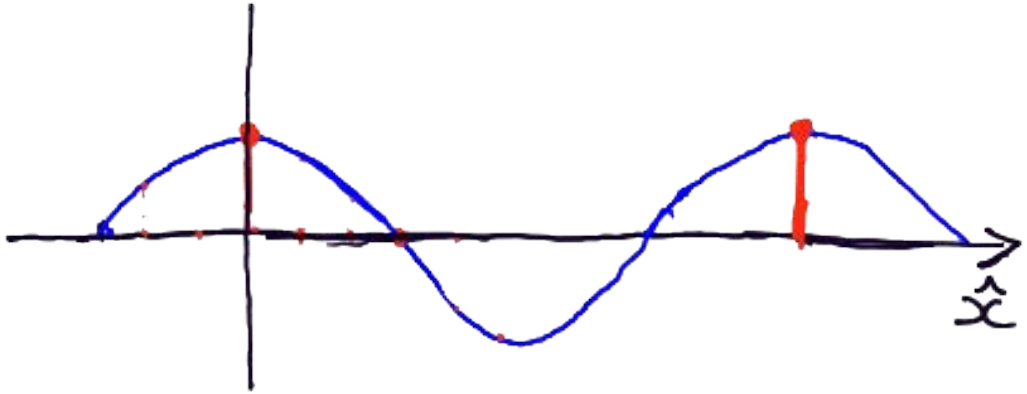
IDEA: How frequently do you need to sample?



LOOKS RIGHT ✓

sampling:

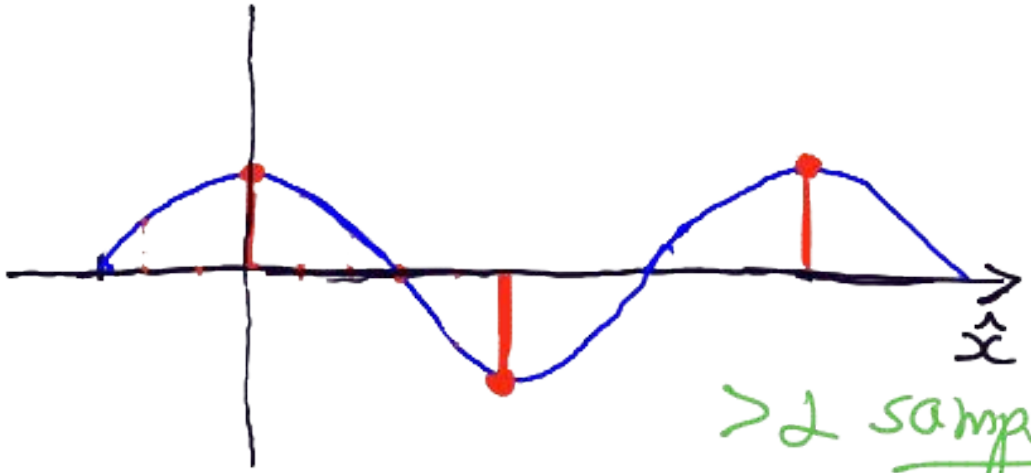
IDEA: How frequently do you need to sample?



Too slow!
(Looks constant)

SAMPLING

#1 is right, but seems over excessive
#2 is too slow
What is the minimum rate?



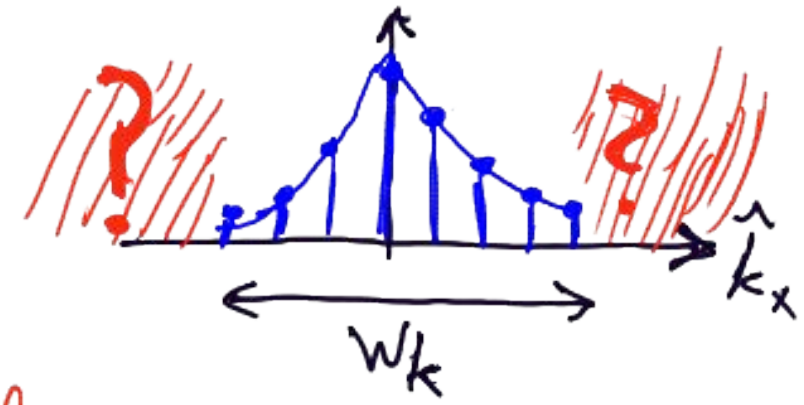
>2 sample period

In general rate >2·BW

show aliasing

sampling in k-space

In MRI $F(k)$ is sampled over a limited range



Problems:

Now → ① DISCRETE SAMPLING

② FINITE SUPPORT

sampling in k-space

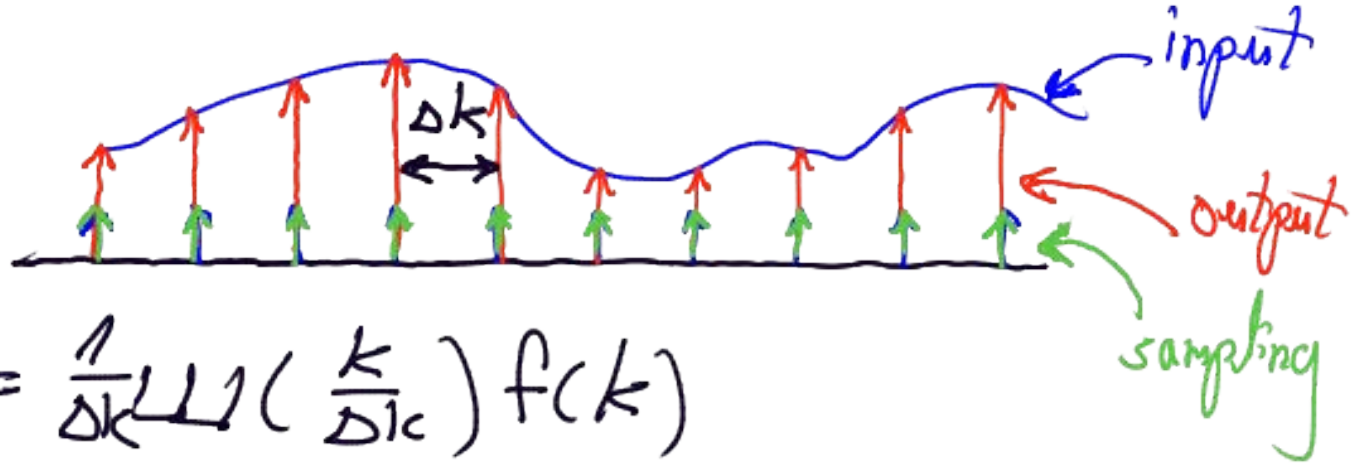
Recall:

$$f(k)\cdot\delta(k-k_0) = f(k_0)\delta(k-k_0)$$

1 sample @ $f(k_0)$

Model sampling using impulse train:

$$\left[\sum_{k=-\infty}^{\infty}\delta(k-n\Delta k)\right] f(k) =$$



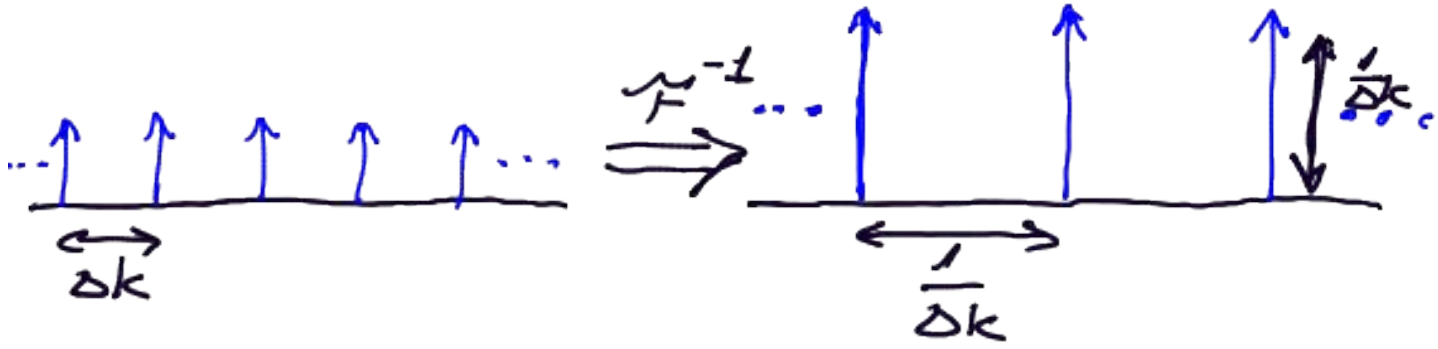
$$= \frac{1}{\Delta k}\amalg\left(\frac{k}{\Delta k}\right) f(k)$$

Recall:

$$F\left(\frac{k}{a}\right) \overset{\mathcal{F}^{-1}}{\Rightarrow} |a| f(ax)$$

So,

$$\frac{1}{\Delta k} \mathrm{III}\left(\frac{k}{\Delta k}\right) \overset{\mathcal{F}^{-1}}{\Rightarrow} \mathrm{III}(\Delta k x)$$



also recall

$$F(k) \cdot G(k) \overset{\mathcal{F}^{-1}}{\Rightarrow} f(x) * g(x)$$

So, sampled

$$\text{if} \quad \hat{F}(k) = F(k) \cdot \frac{1}{\Delta k} \amalg\left(\frac{k}{\Delta k}\right)$$

$$\Downarrow \mathcal{F}^{-1}$$

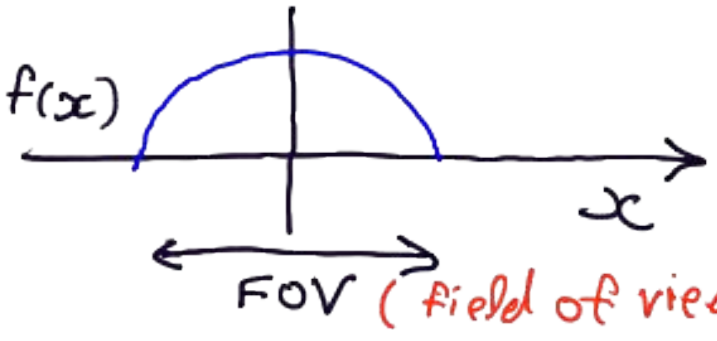
then

$$\hat{f}(x) = \mathcal{F}^{-1}\{\hat{F}(k)\} = f(x) * \amalg(\Delta k x)$$

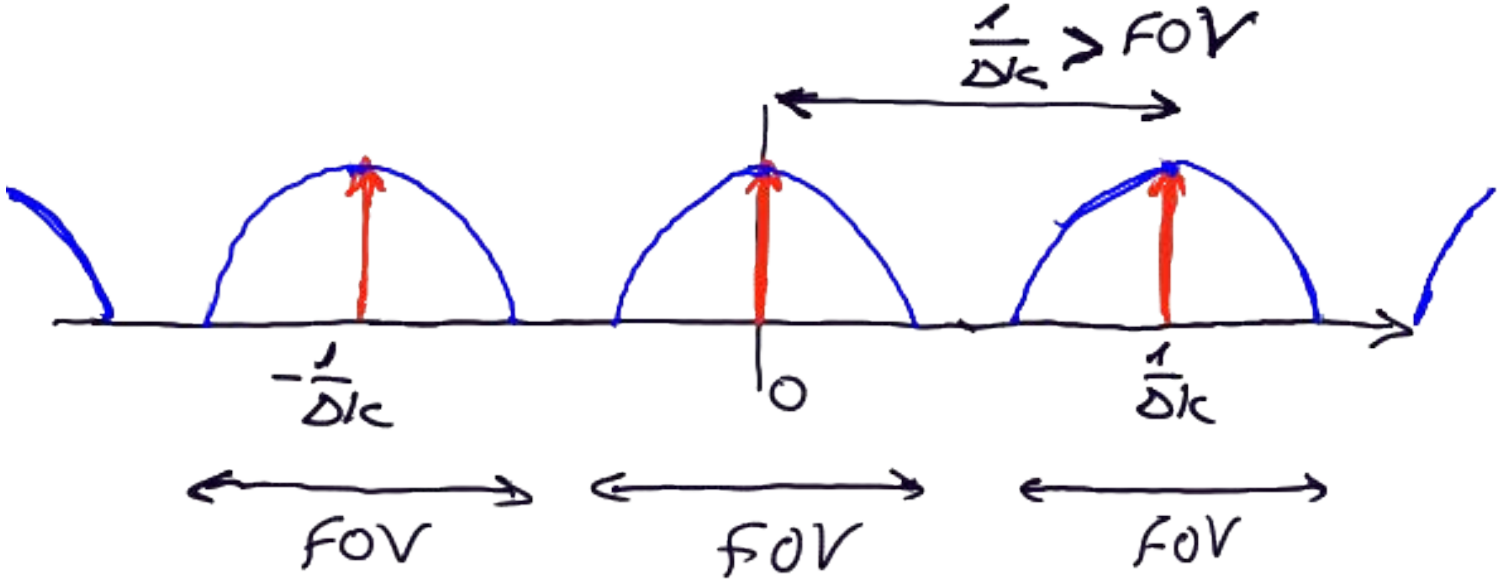
Recall:

$$f(x) * \delta(x - x_0) = f(x - x_0)$$

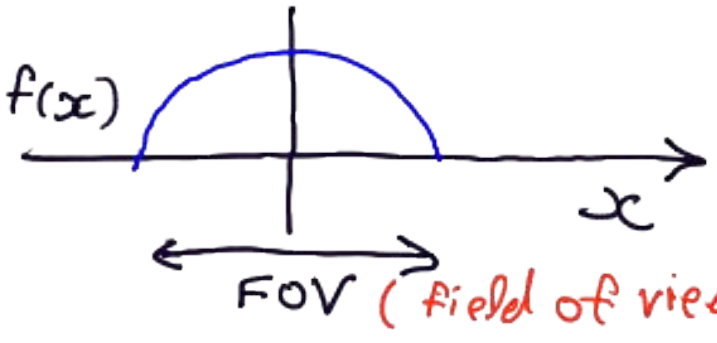
If $f(x)$ is SPACE LIMITED:



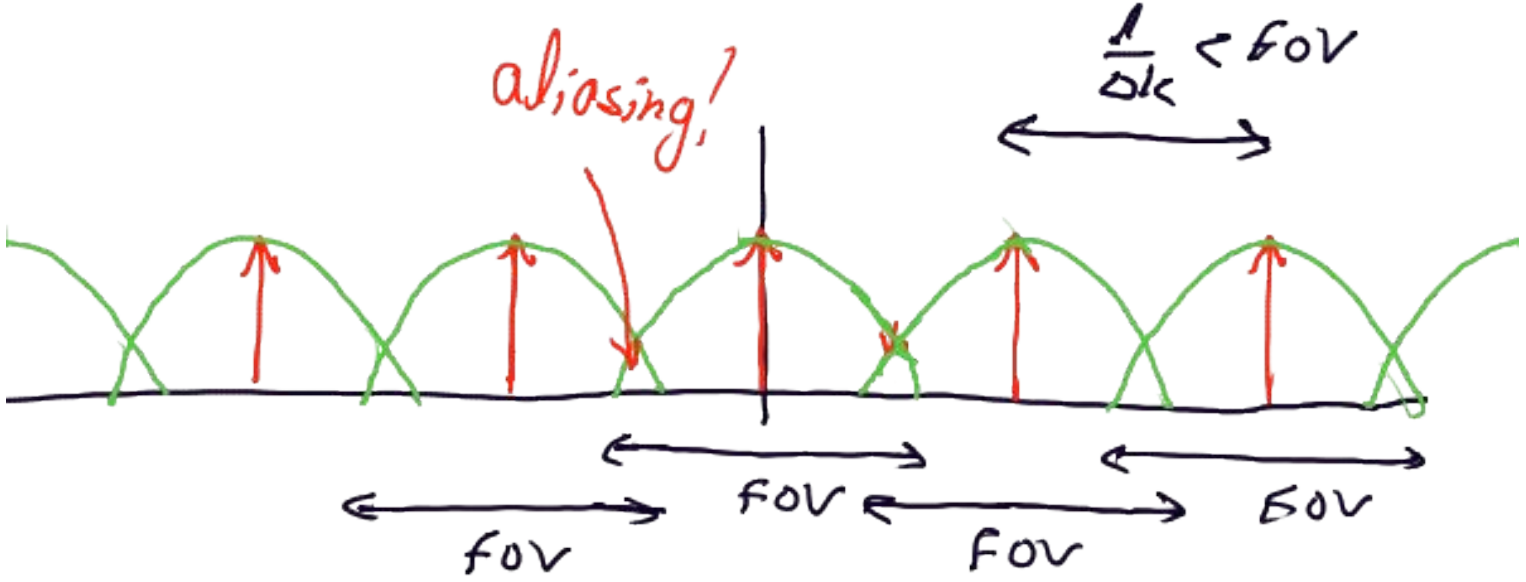
$$\hat{f}(x) = f(x) * \amalg(\Delta k x)$$



If $f(x)$ is SPACE LIMITED:

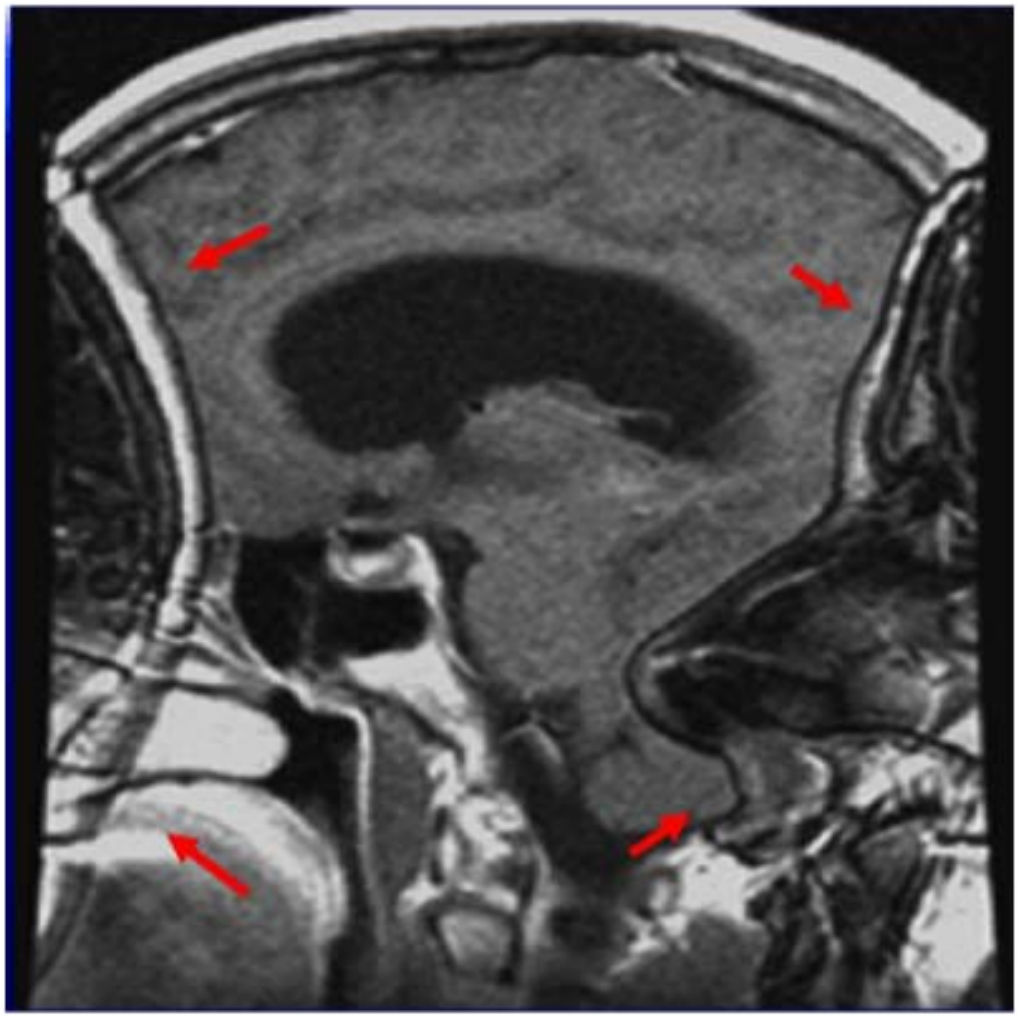


$$\hat{f}(x) = f(x) * \amalg(\Delta k x)$$



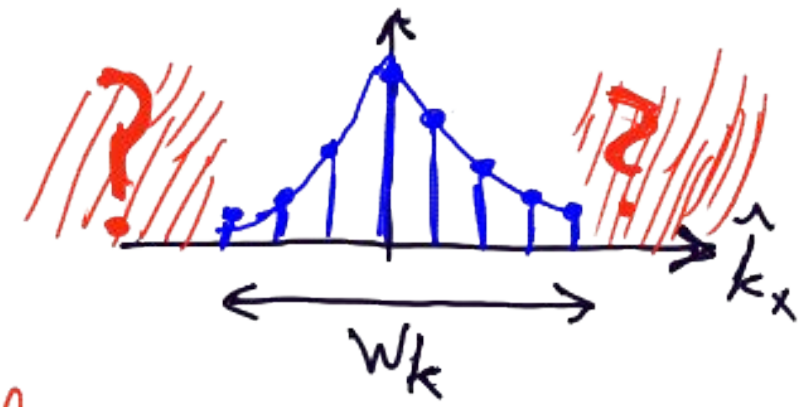
Aliasing in x due to large Δk_x

8a



sampling in k-space

In MRI $F(k)$ is sampled over a limited range



Problems:

① DISCRETE SAMPLING

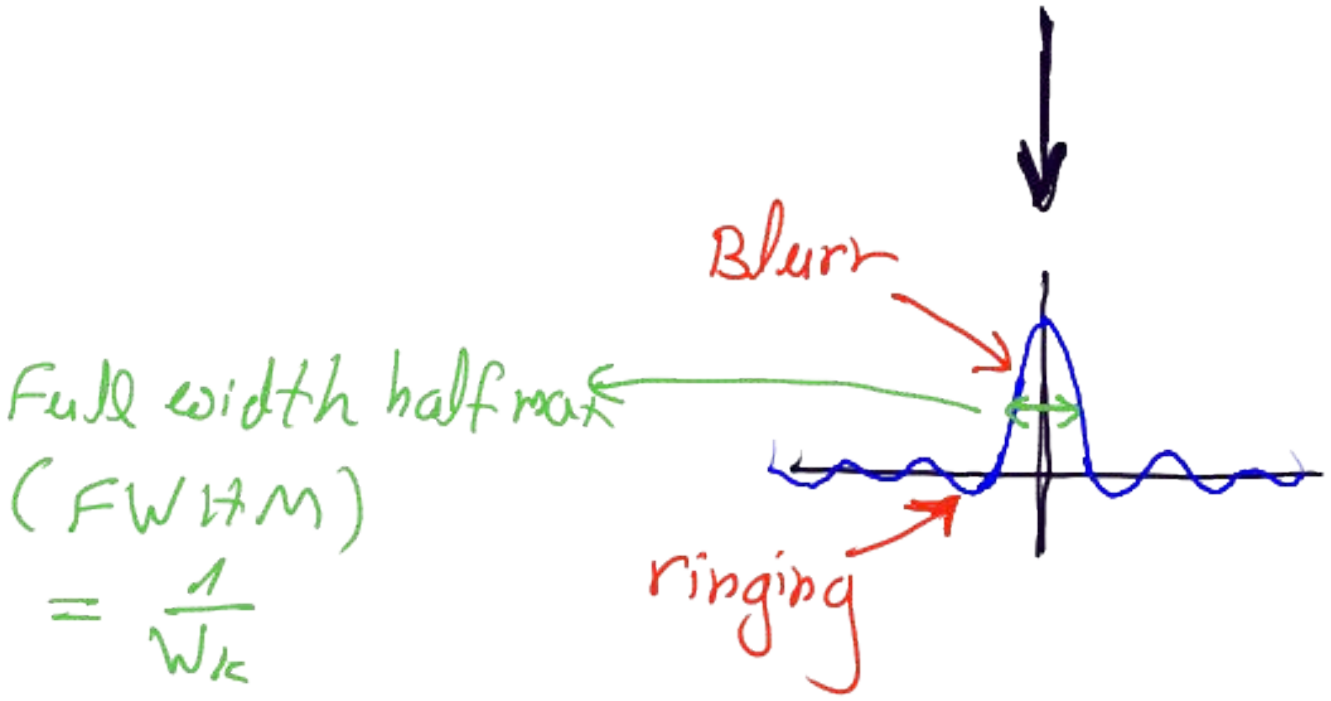
Now → ② FINITE SUPPORT

Finite support sampling

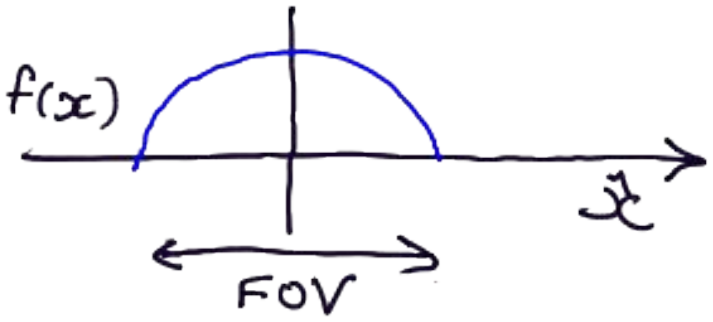
$$F_W(k) = F(k) \sqcap \left(\frac{k}{W_k} \right)$$

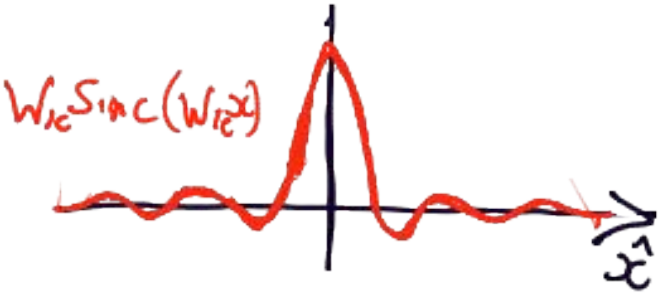
"windowed"

$$\Rightarrow f_W(x) = f(x) * W_k \operatorname{sinc}(W_k x)$$

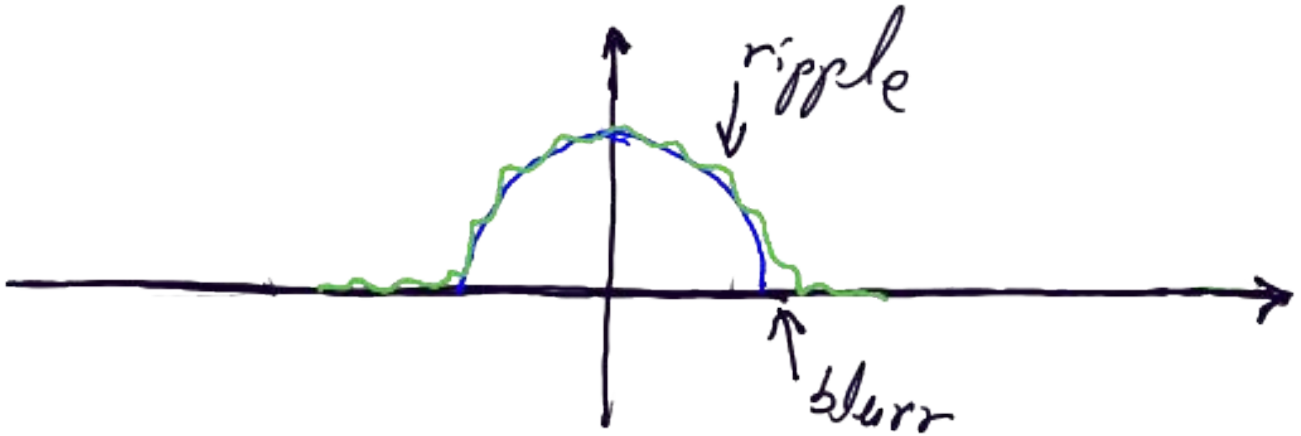


If $f(x)$ is



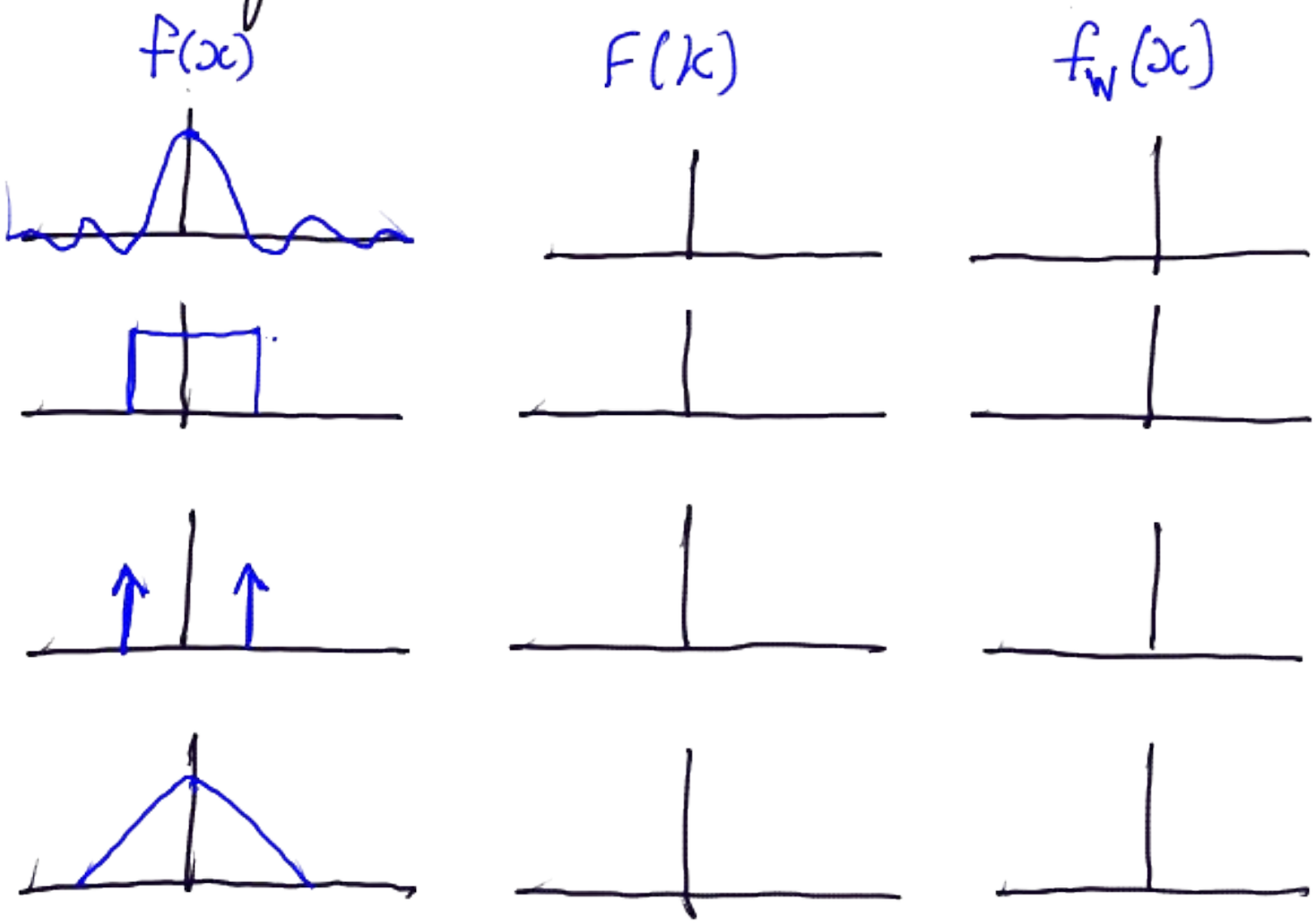


$$f_W(x) = f(x) * W_c \mathrm{sinc}(W_c x)$$



QUIZ

Rank the "strength" of artifact due to k-space truncation



Discrete & finite sampling

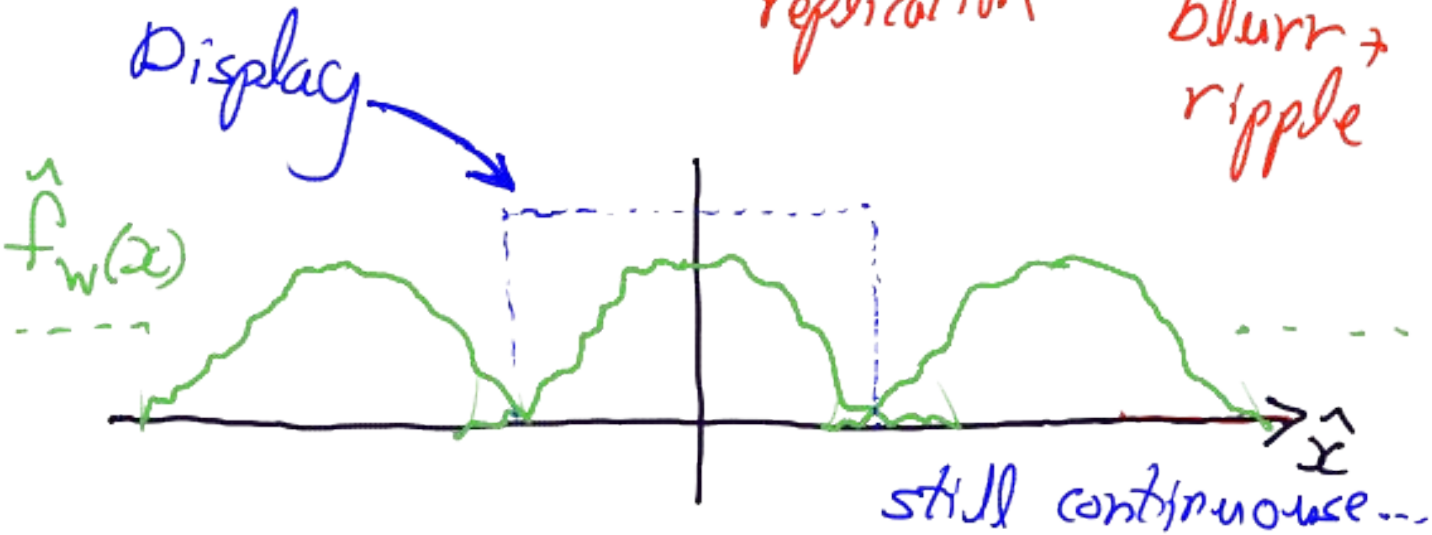
Together:

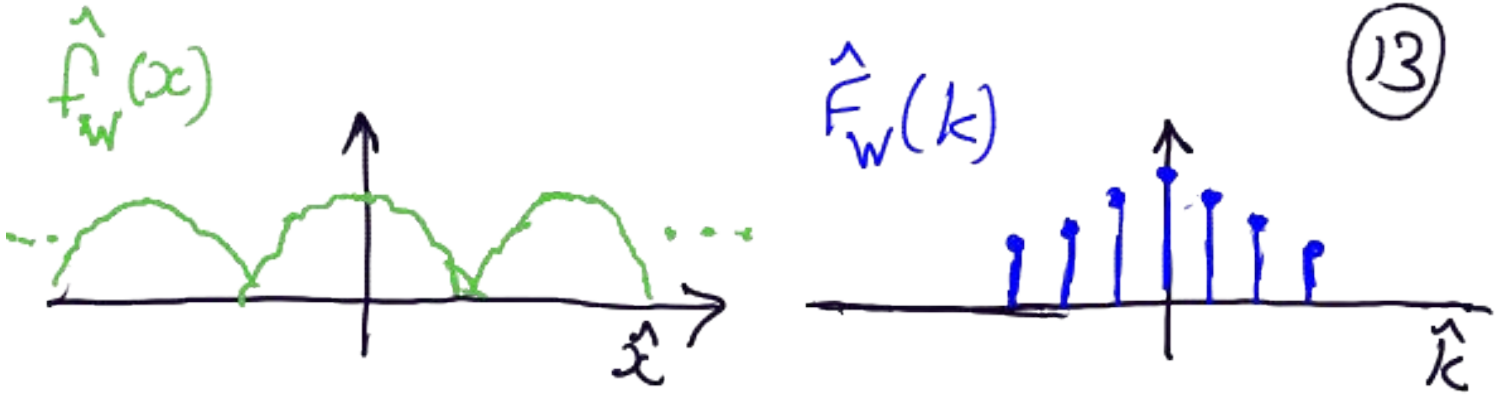
$$\hat{F}_W(k) = F(k) \cdot \frac{1}{\Delta k} \amalg\left(\frac{k}{\Delta k}\right) \Pi\left(\frac{k}{W_k}\right)$$

$$\xrightarrow{\mathcal{F}^{-1}} \quad \hat{f}_W(x) = f(x) * \amalg(\Delta k x) * W_k \operatorname{sinc}(W_k x)$$

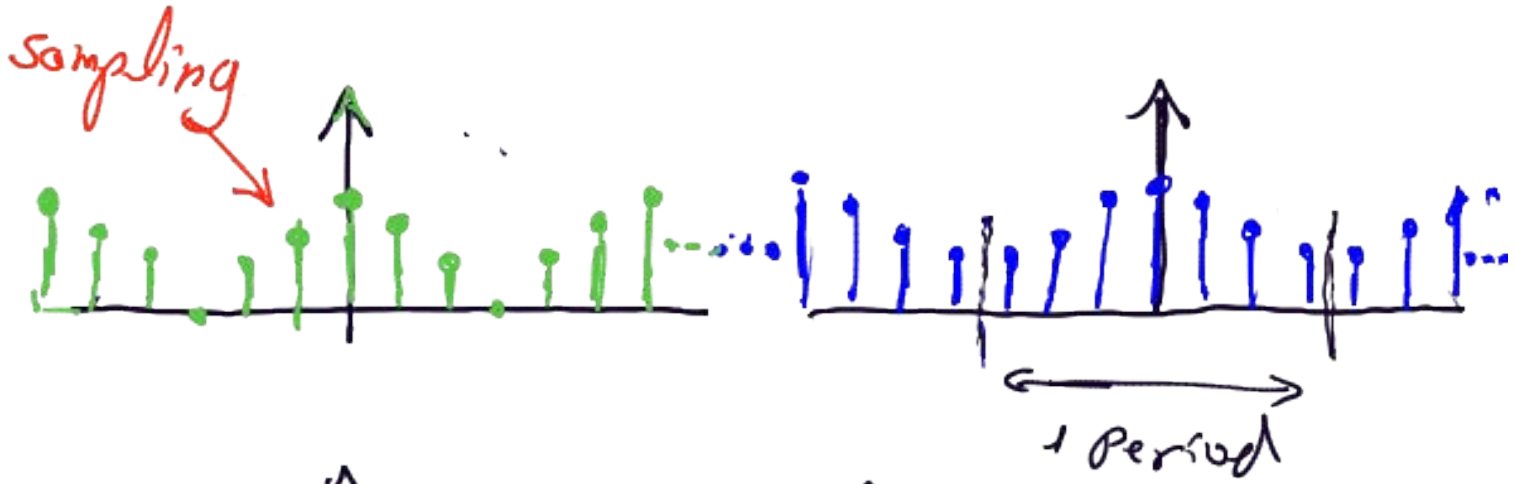
replication

blurr + ripple





...But images are discrete pixels...
so $\check{f}_w(x)$ is sampled



$$\check{f}_w(x) \Longleftrightarrow \check{\hat{F}}_w(k)$$

$$f_w[n] \overset{DFT}{\Longleftrightarrow} F_w[m]$$

integer

DFT (centered)

$$F[m] = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} f[n]\, e^{-i 2\pi \frac{mn}{N}}$$

$$f[n] = \frac{1}{N} \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} F[m]\, e^{+2\pi \frac{mn}{N}}$$

FAST computation with FFT

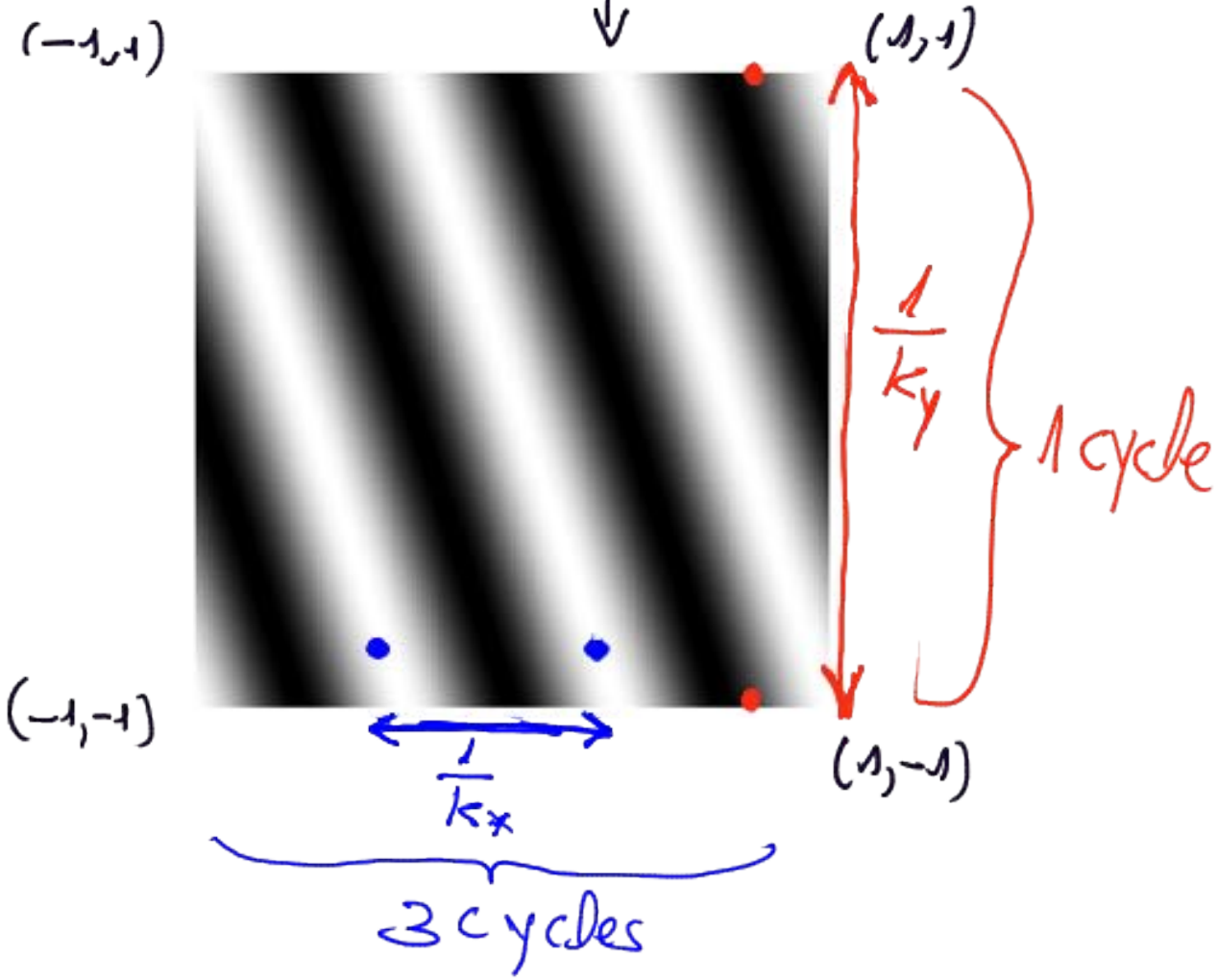
MATLAB:

DFT = fftshift(fft(fftshift))

IDFT = fftshift(ifft(ifftshift))

SPATIAL HARMONICS & 2D FT

This is a spatial harmonic



$$f(x,y) = \cos(2\pi(3x+y))$$

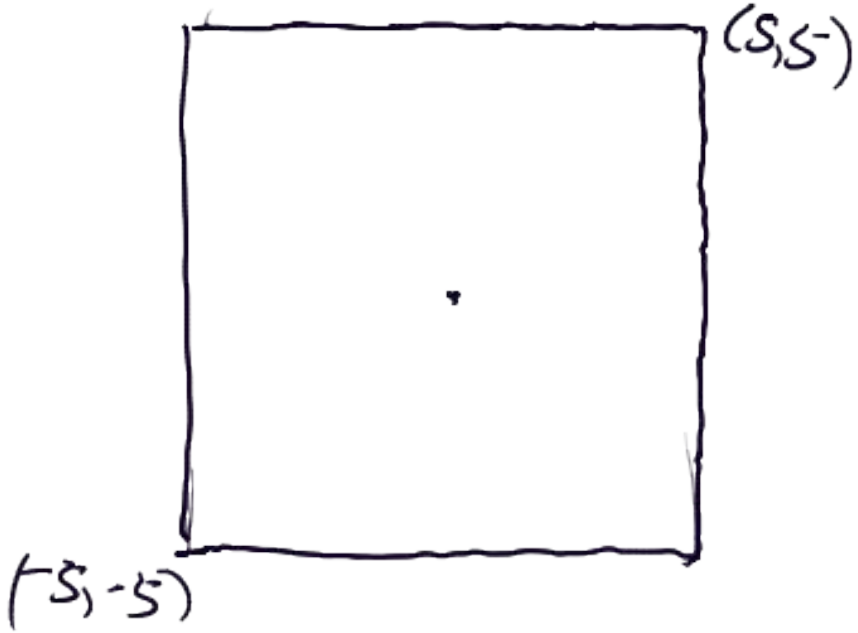
2D FOURIER TRANSFORMS

$$F(k_x,k_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-i2\pi(k_x x+k_y y)}\,dx\,dy$$

Q. what is the 2D-FT of

$$f(x,y)=\cos(2\pi(3x+y))$$

A:



seperable functions

$$f(x,y) = f(x)f(y)$$

Q. Which is the 2D-FT?

A. (1) $F(k_x) * F(k_y)$

(2) $F(k_x) \cdot F(k_y)$

sampling & truncation

$$\hat{F_w}(k_x,k_y) = F(k_x,k_y)\cdot\frac{1}{\Delta k_x \Delta k_y}\amalg\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)\cdot \Pi\left(\frac{k_x}{W_x},\frac{k_y}{W_y}\right)$$

aliasing

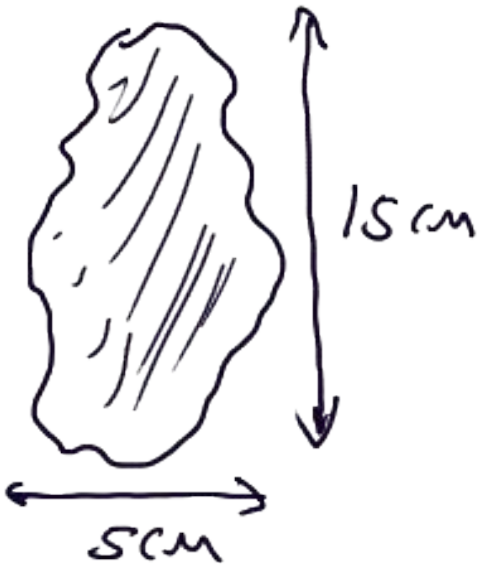
$$\hat{f}_w(x,y) = f(x,y) ** \amalg(\Delta k_x x, \Delta k_y y) ** W_x W_y \operatorname{sinc}(W_x x, W_y y)$$

Blurr

more degrees of freedom for sampling!

2D SAMPLING EXAMPLE

GIVEN $f(x,y)$:

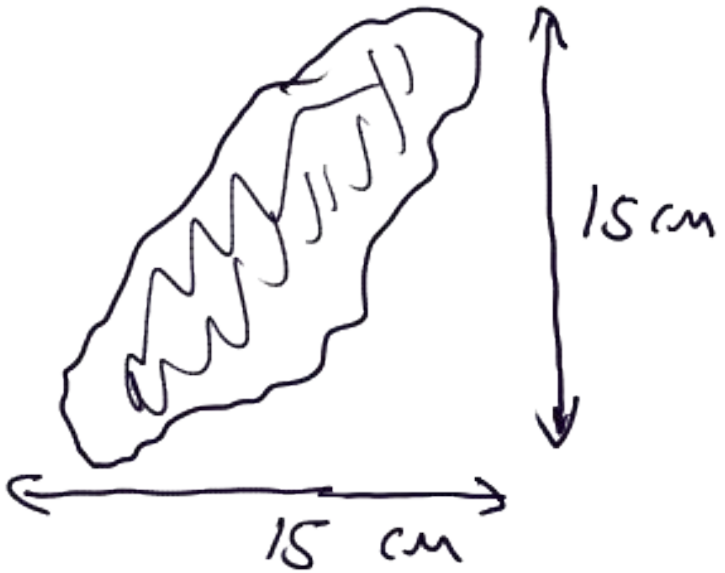


what is Δk_x, Δk_y?

. (to avoid aliasing)

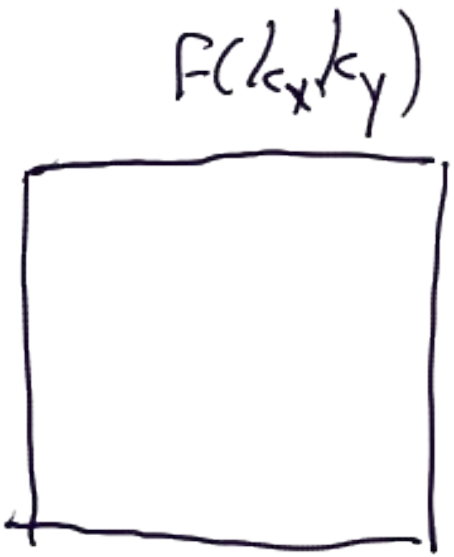
2D SAMPLING EXAMPLE

GIVEN $f(x,y)$:



what is Δk_x, Δk_y ?

. (to avoid aliasing)



CIRCULARLY SYMMETRIC

$$f(x,y) = f(r)$$

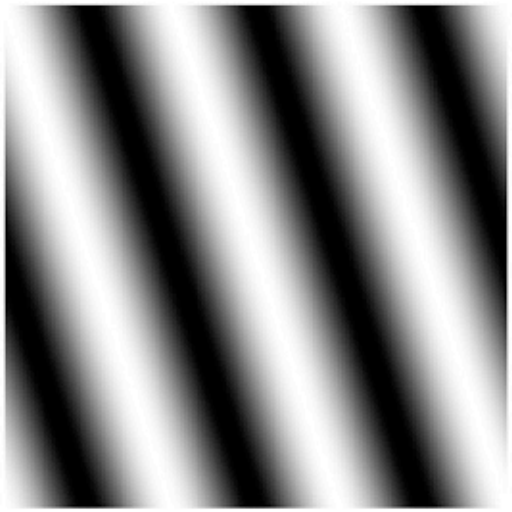
$$\mathcal{F}_{2D}\{f(r)\} = 2\pi \int_0^\infty r f(r) J_0(2\pi r \rho_k)\, dr$$

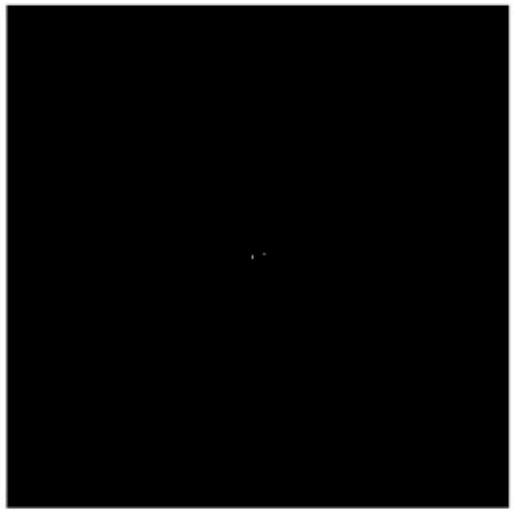
Hankel xform of zero'th order

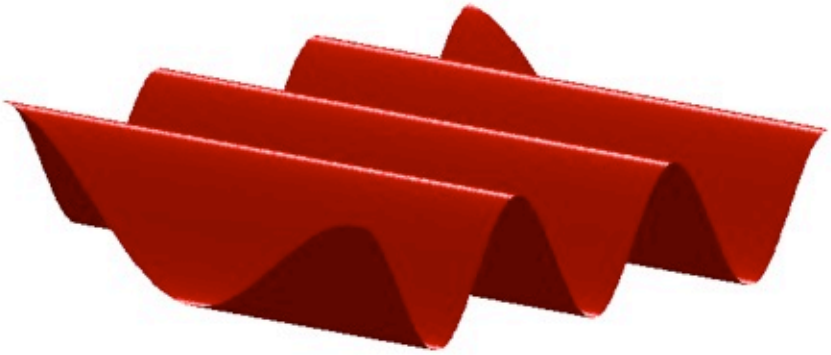
.... also circularly symmetric

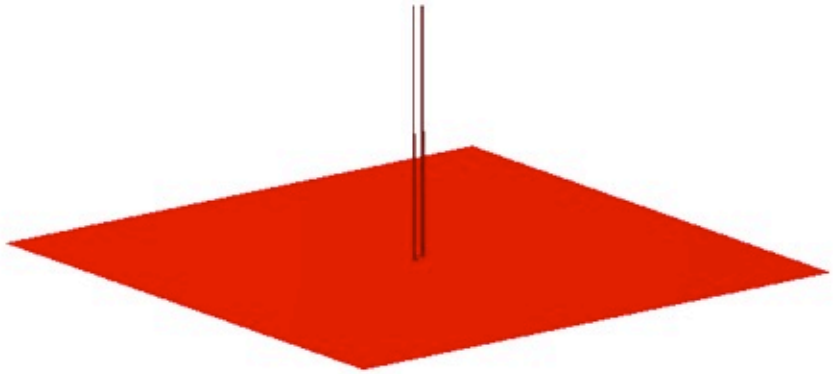
$$\Pi(r) \Longrightarrow \frac{J_1(\pi \rho_k)}{2\rho_k} \triangleq \mathrm{jinc}(\rho_k)$$

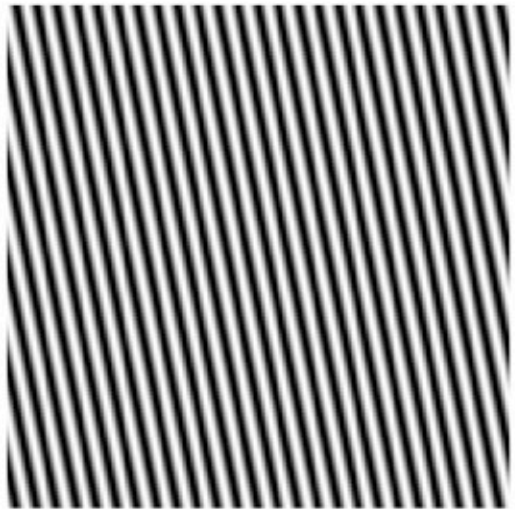
like a sinc — but different

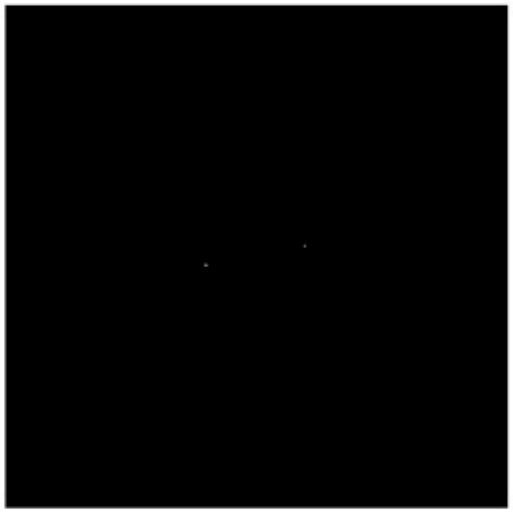


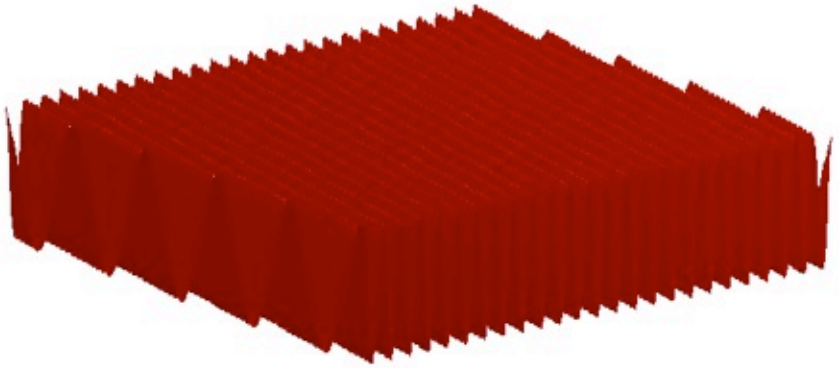


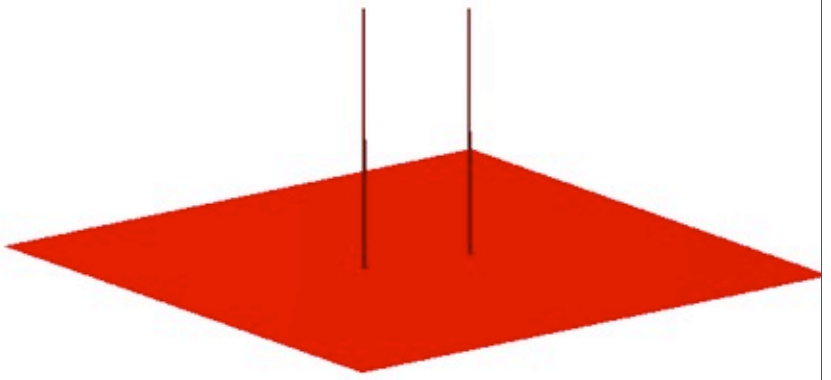


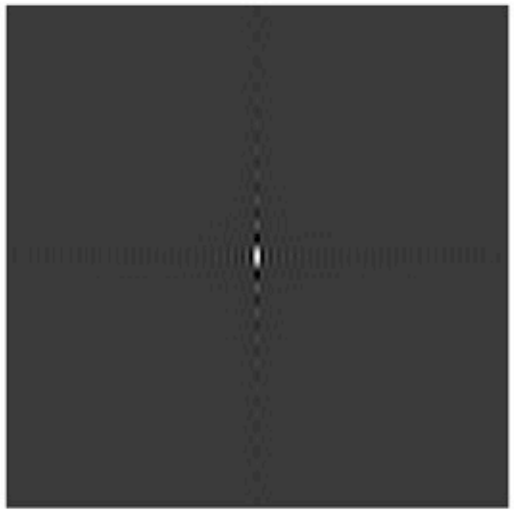


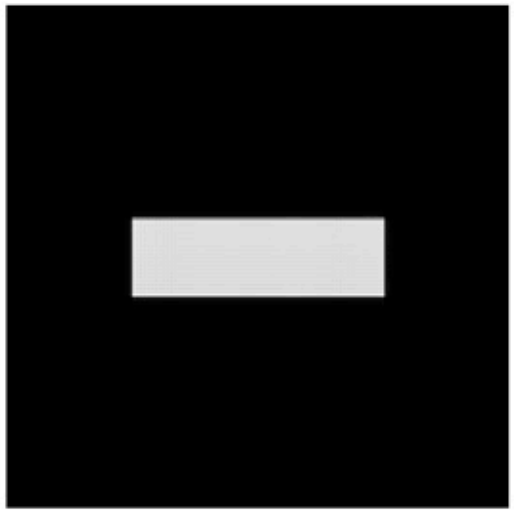


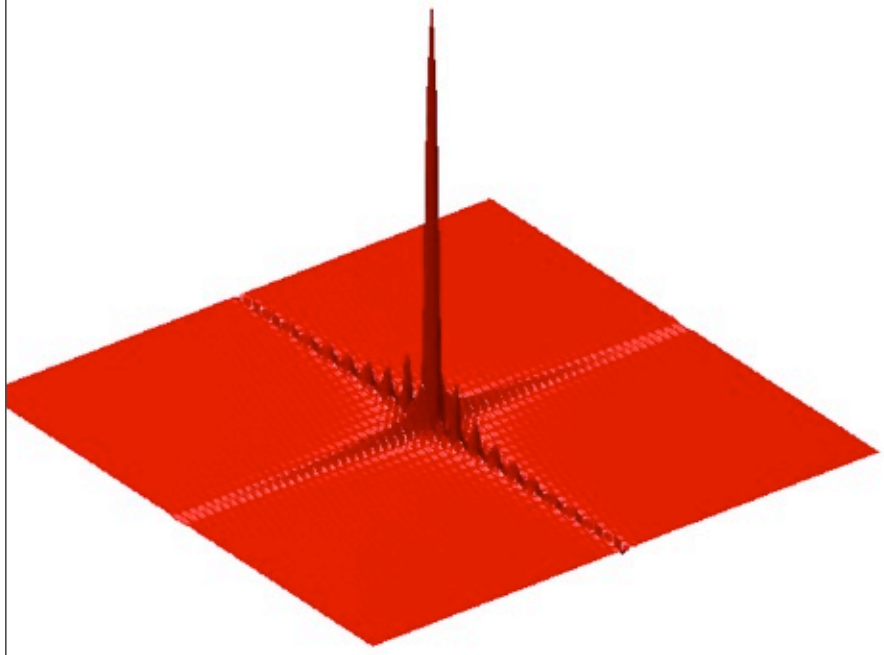


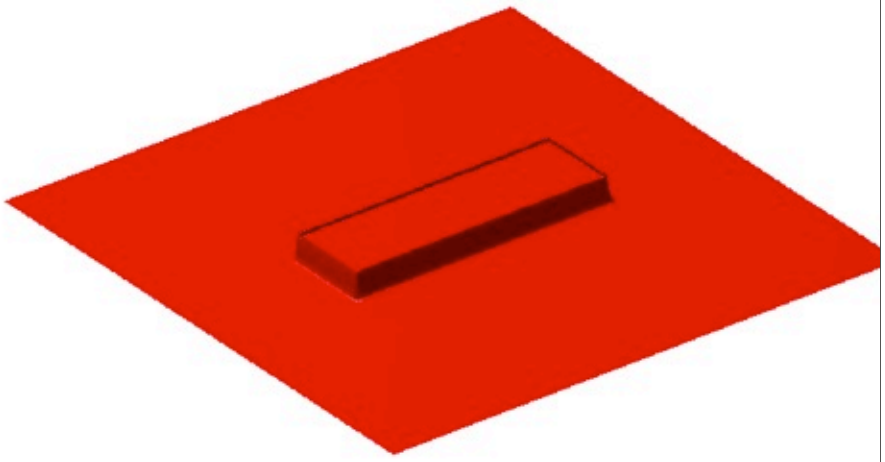


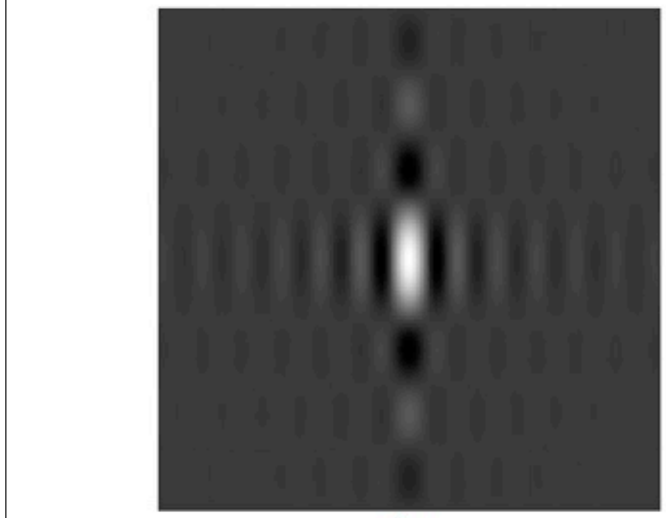


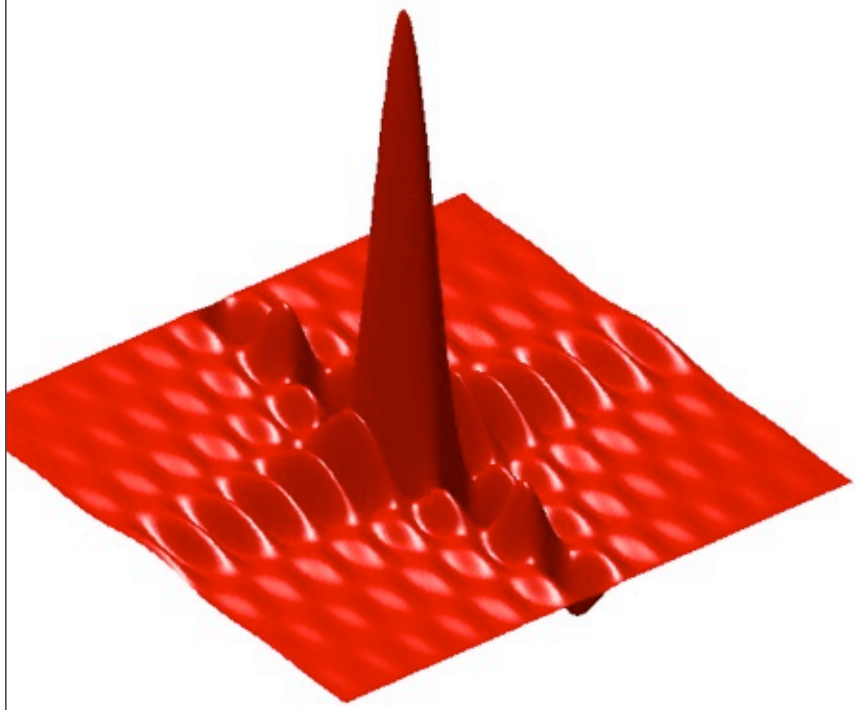


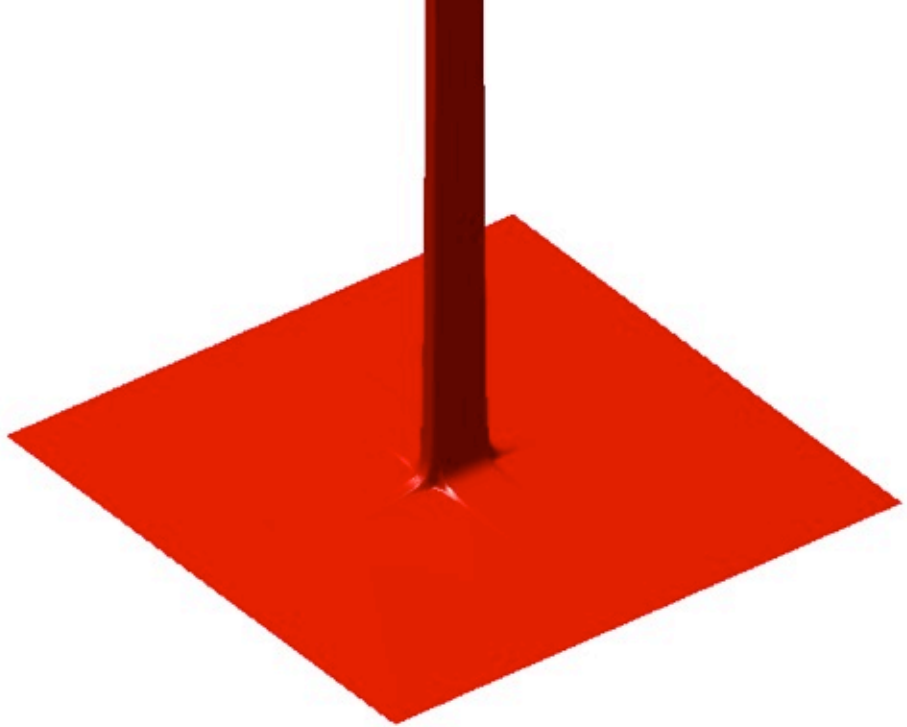


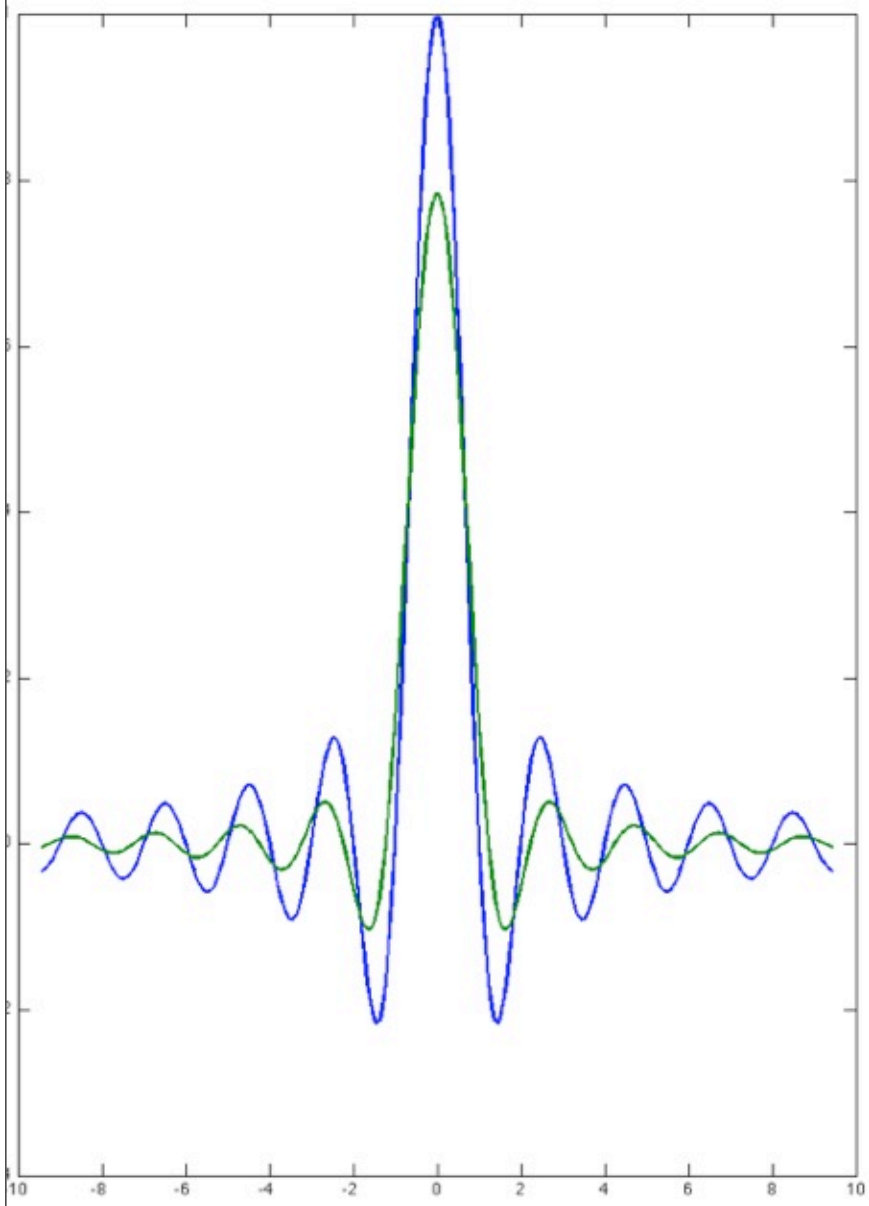


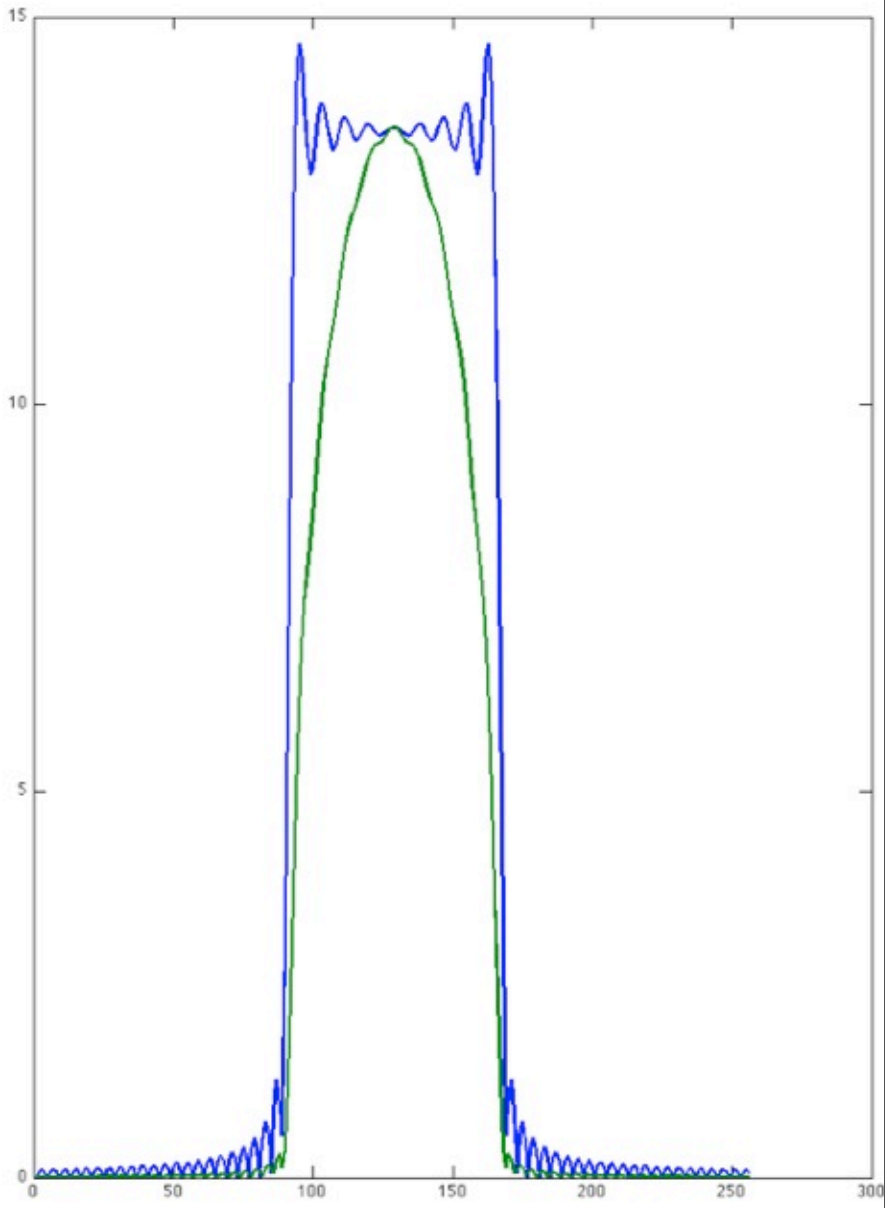




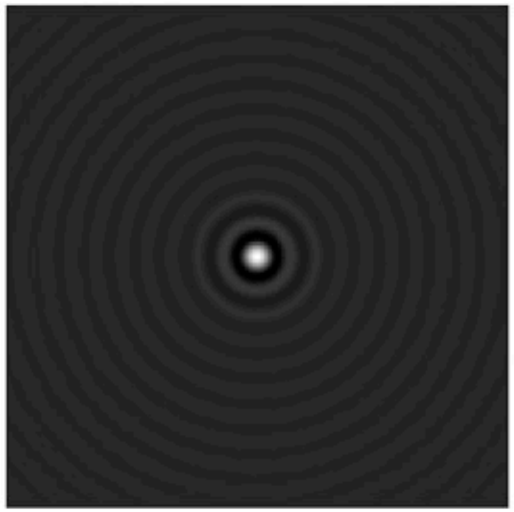


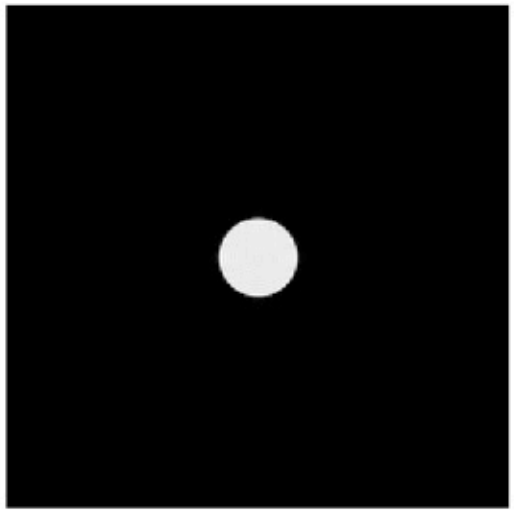


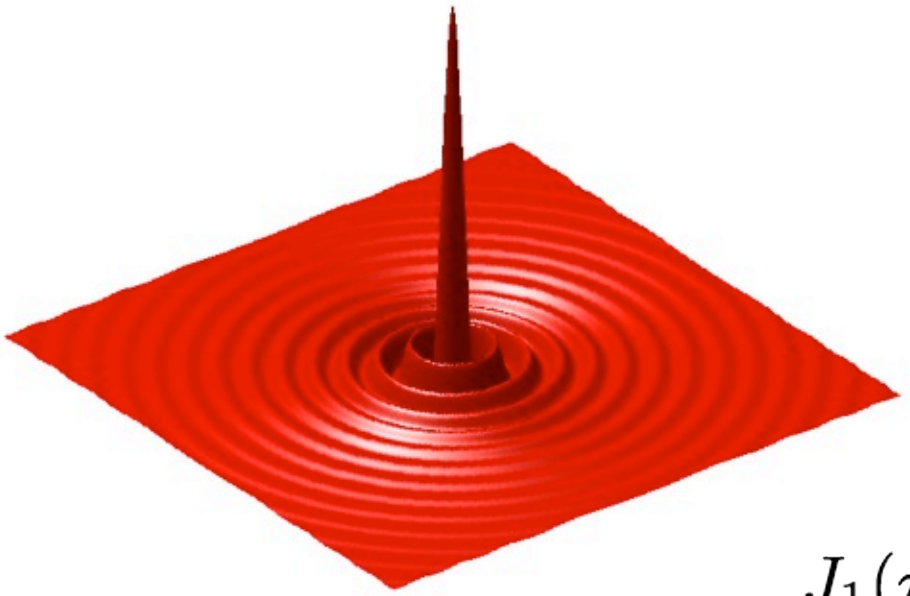


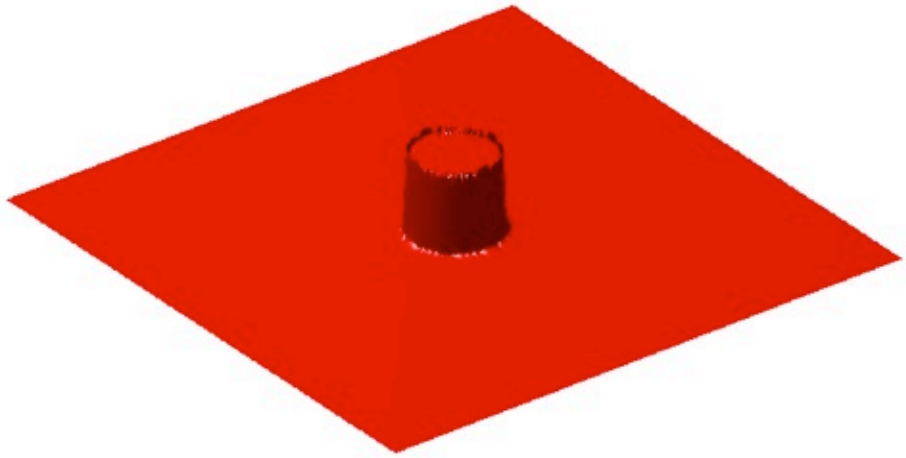


$$\frac{J_1(\pi x)}{2x} \qquad \frac{\sin(\pi x)}{\pi x}$$

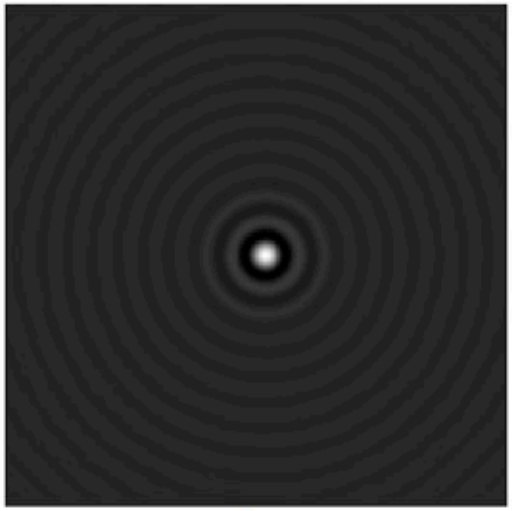


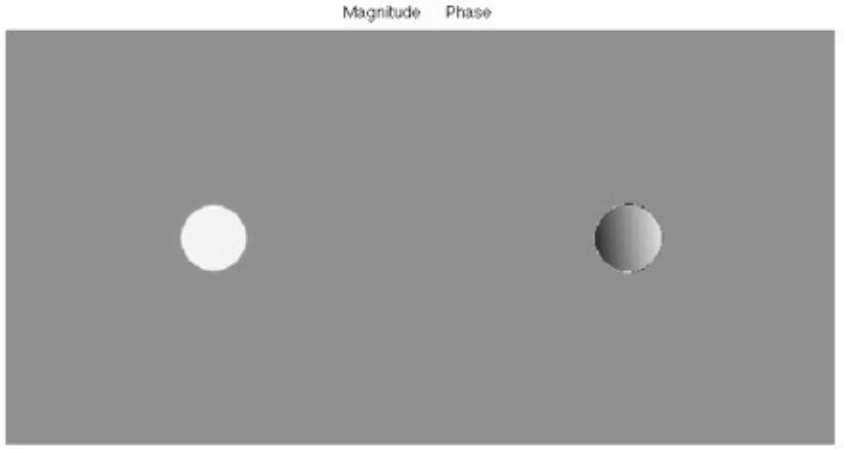
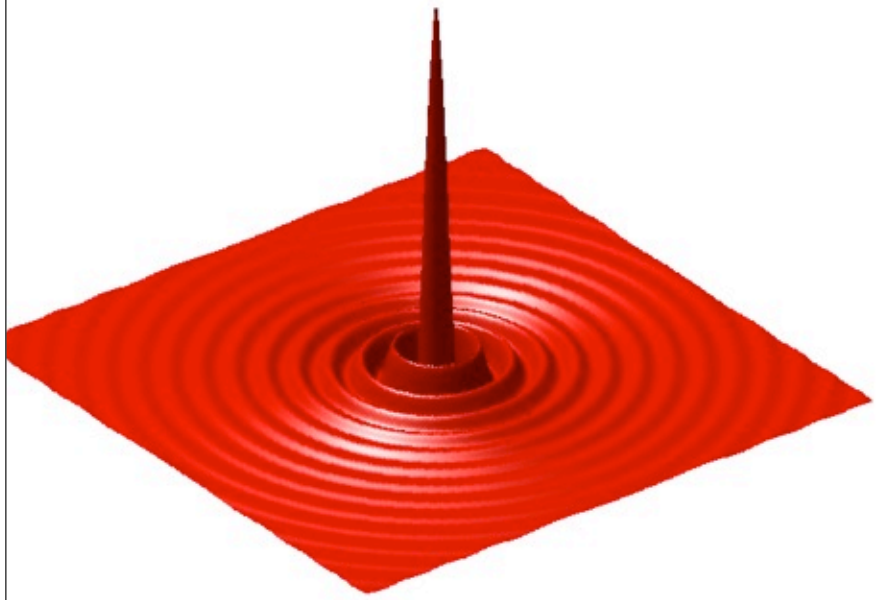
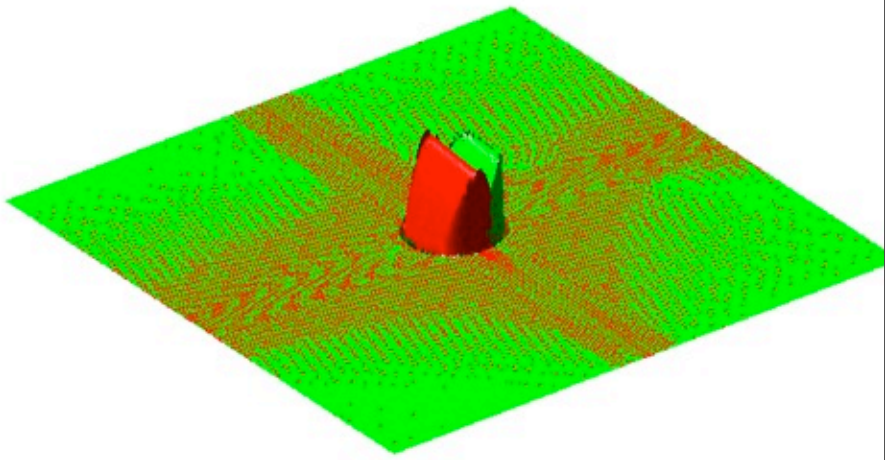
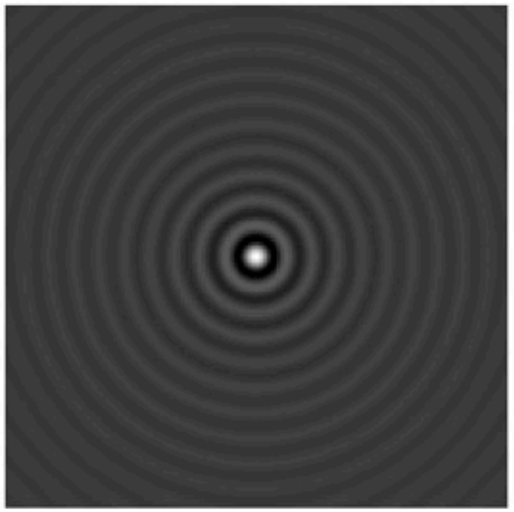
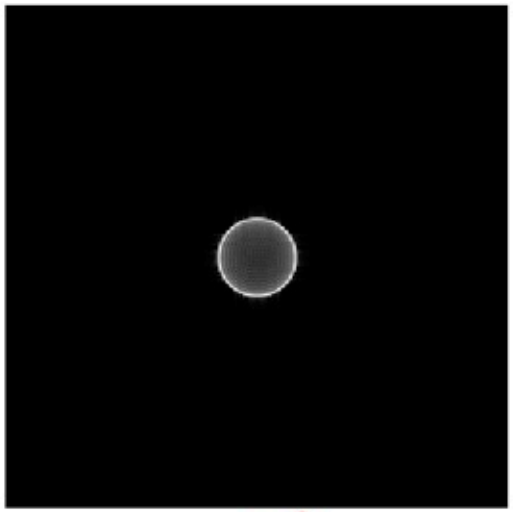
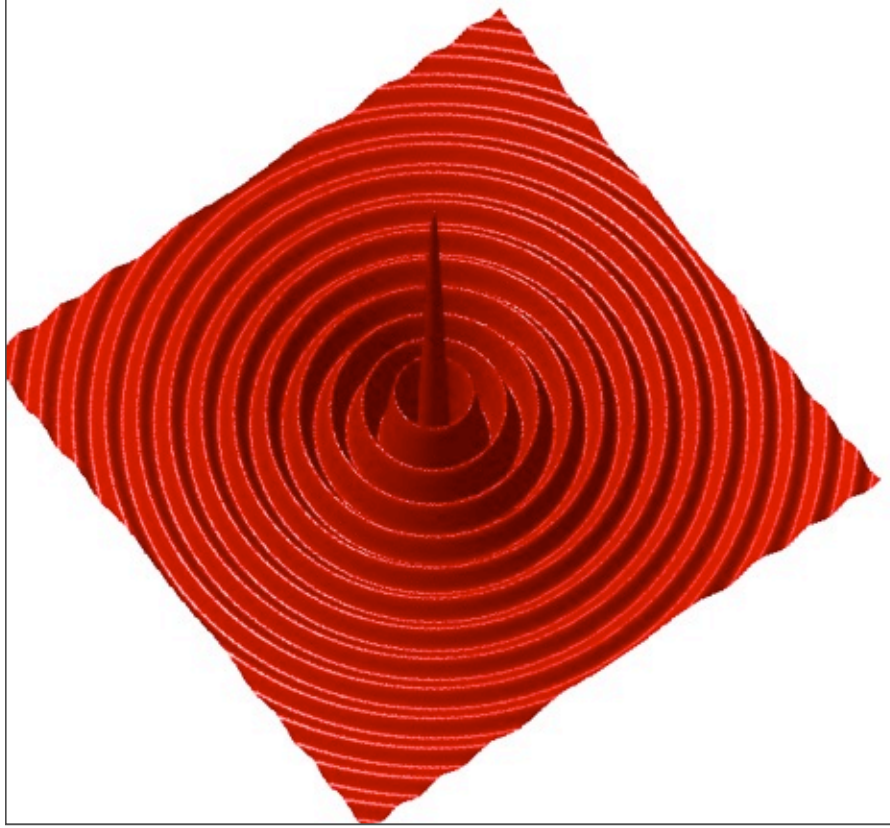
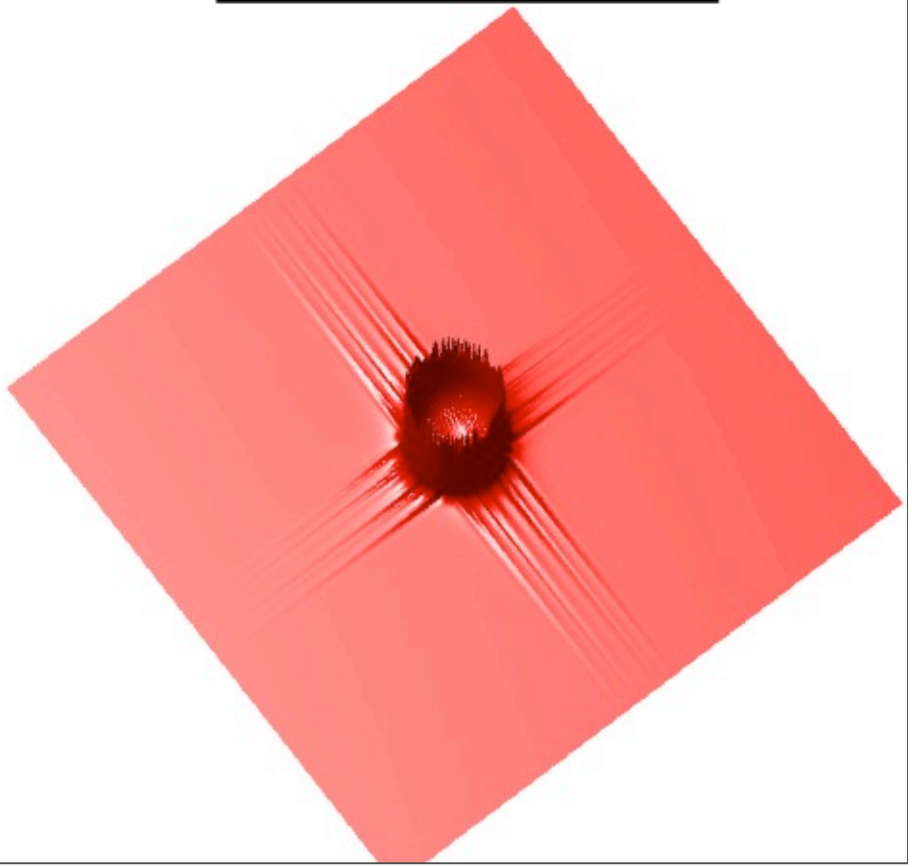
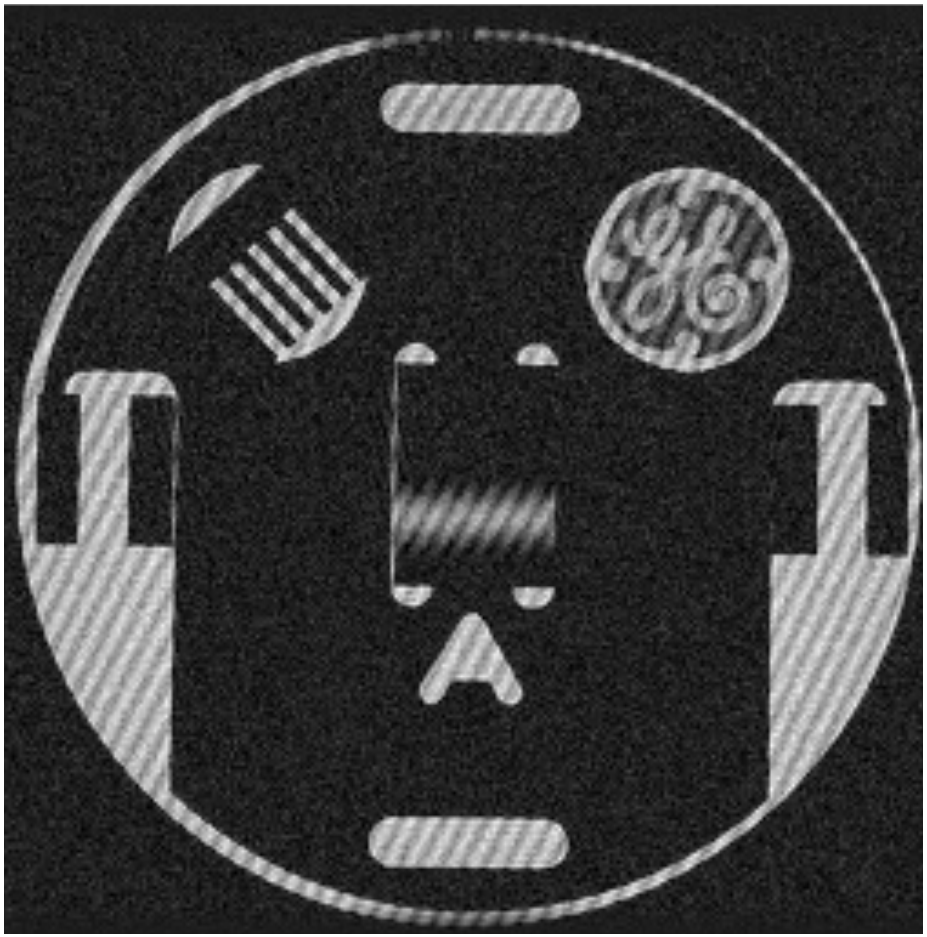
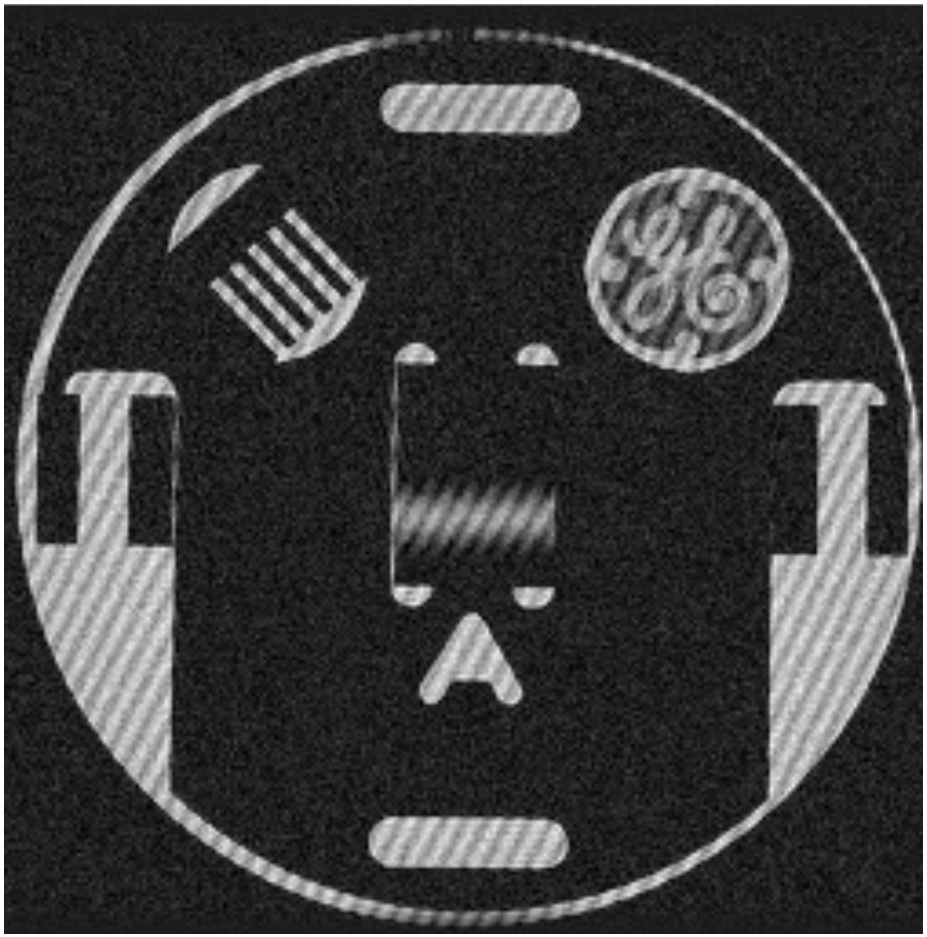
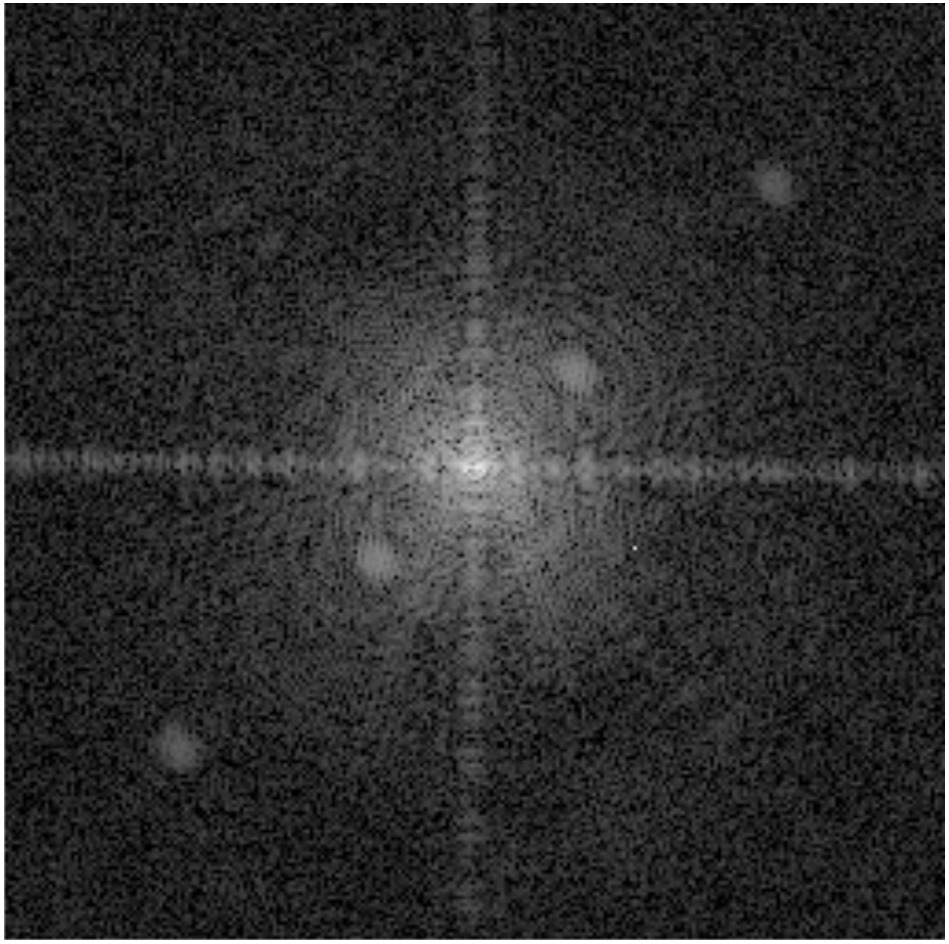
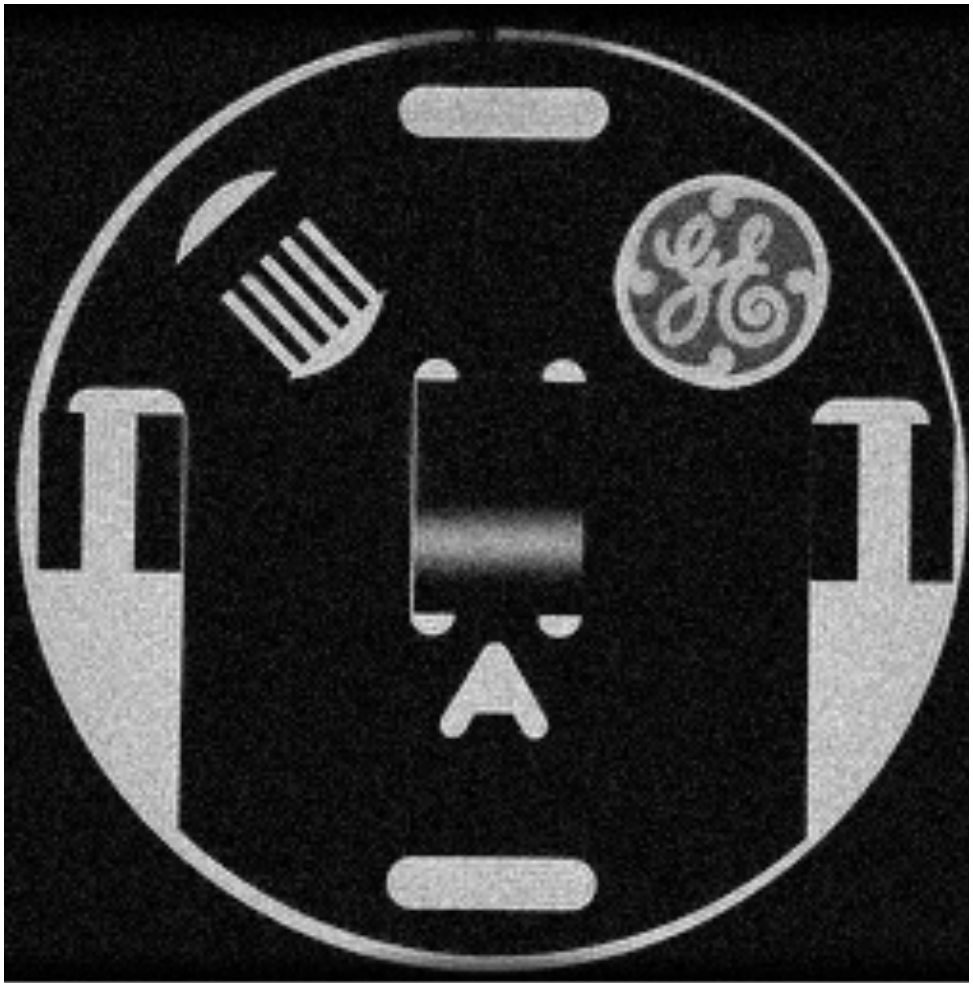
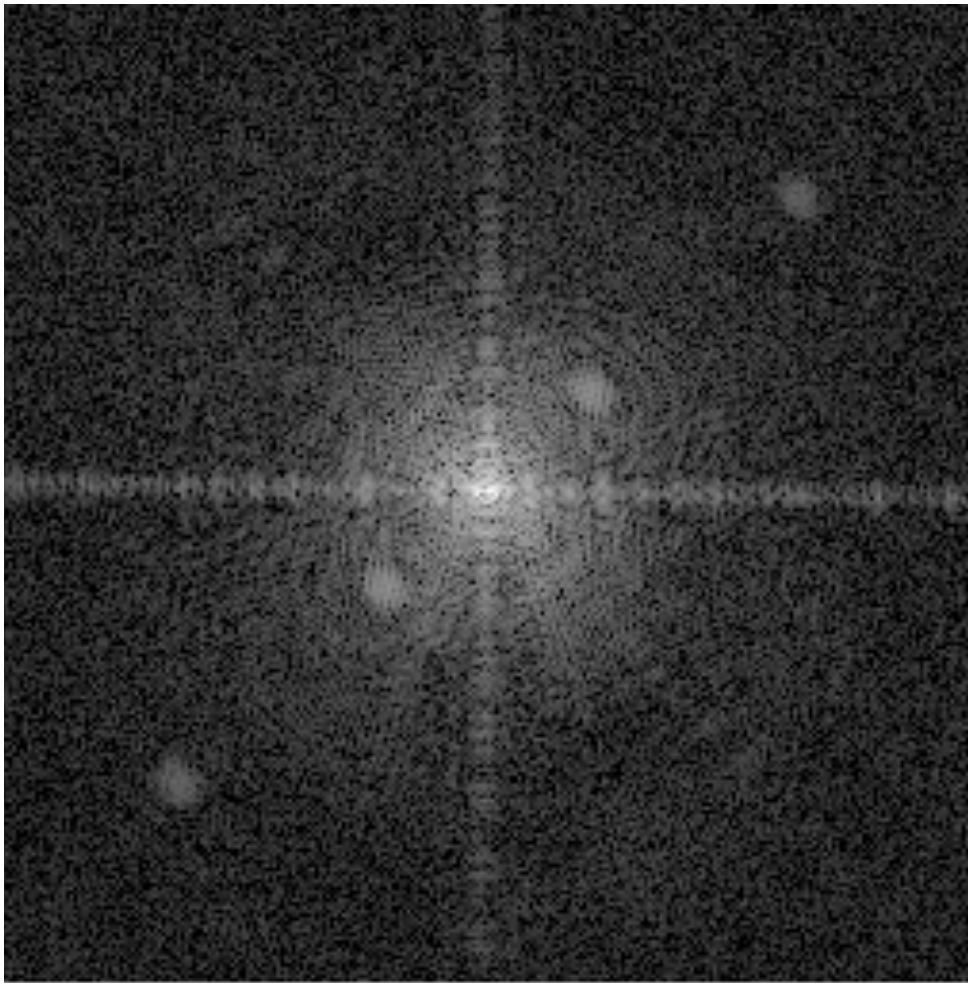
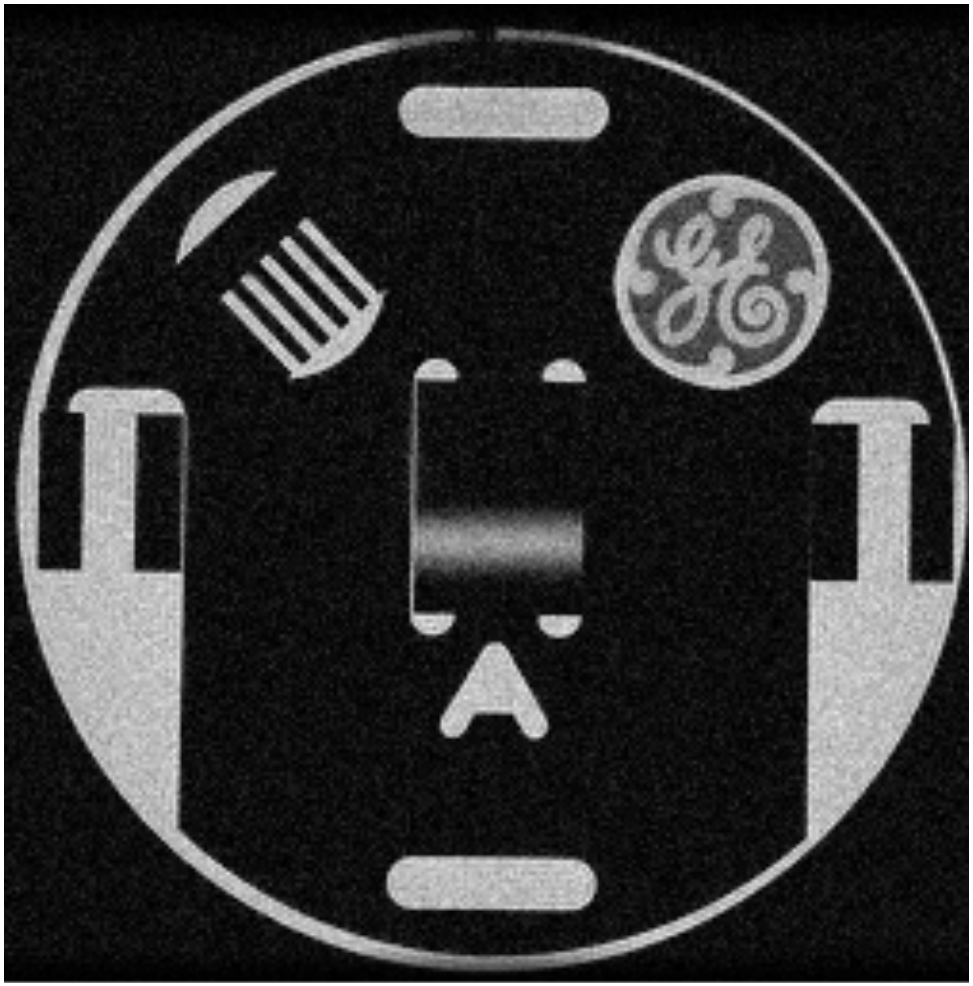
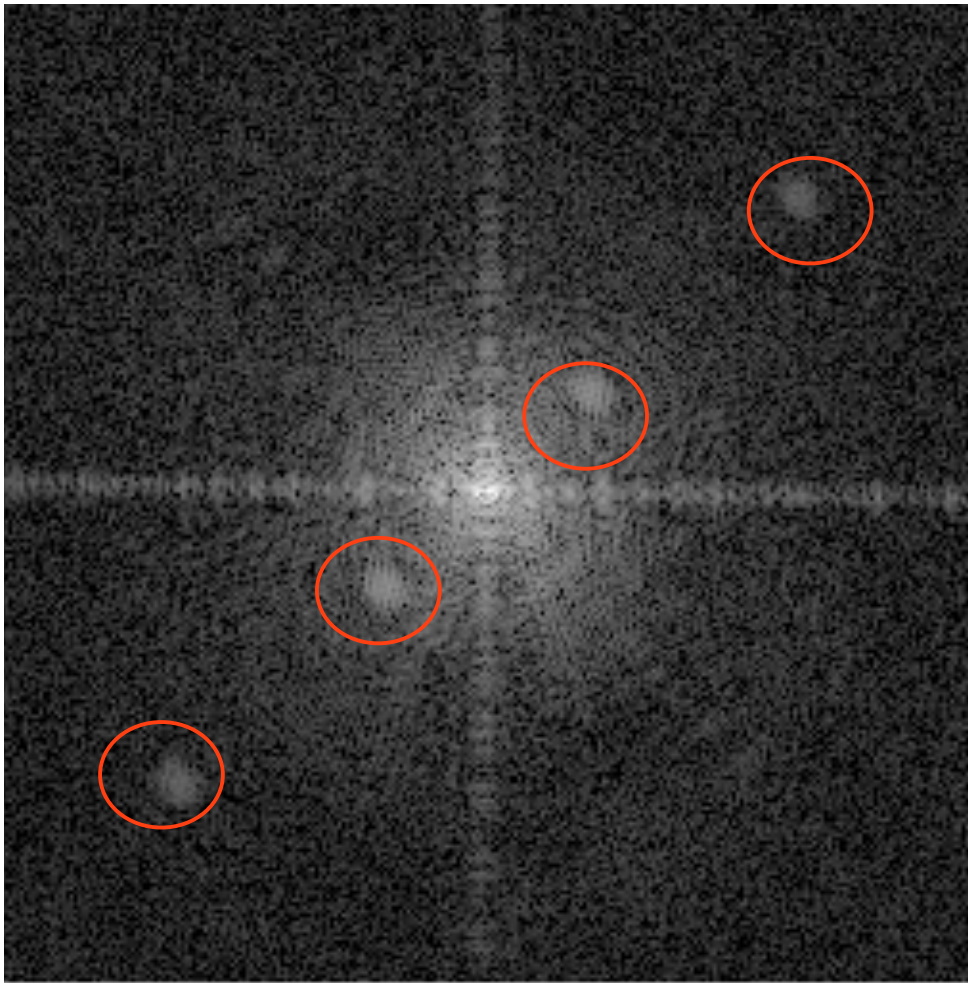
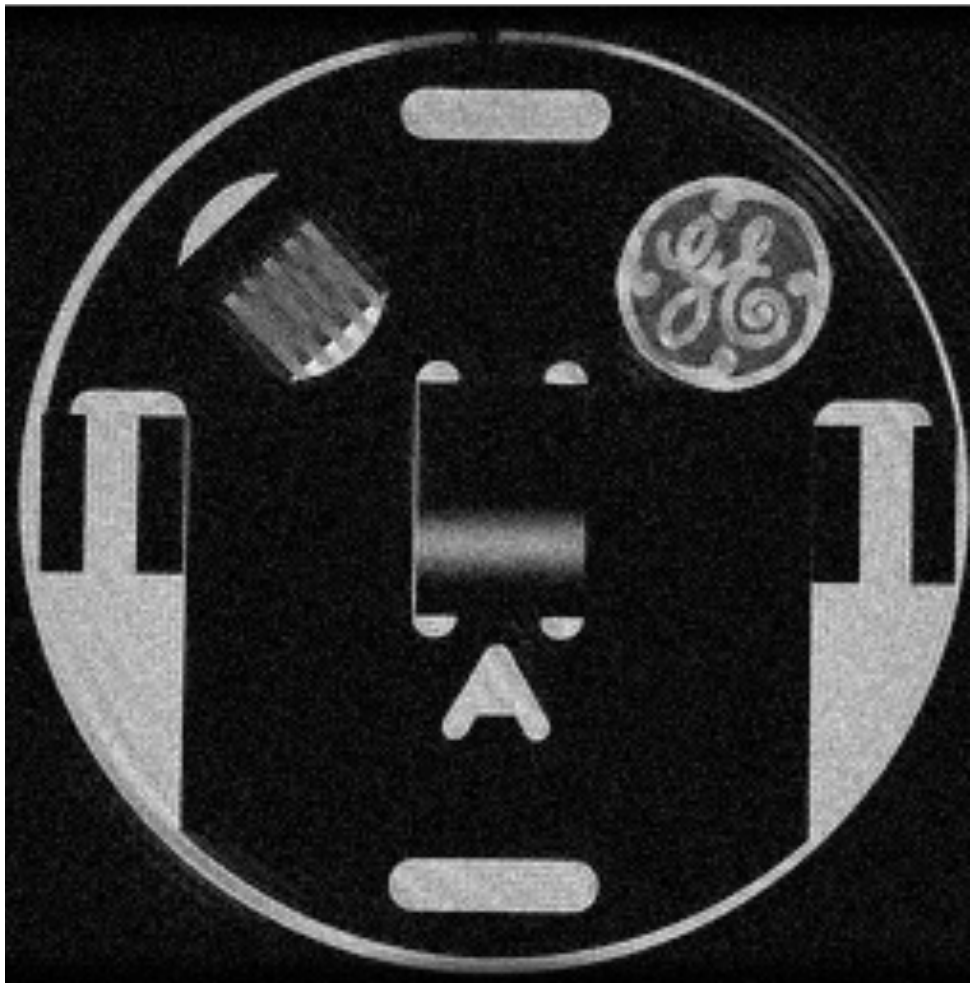
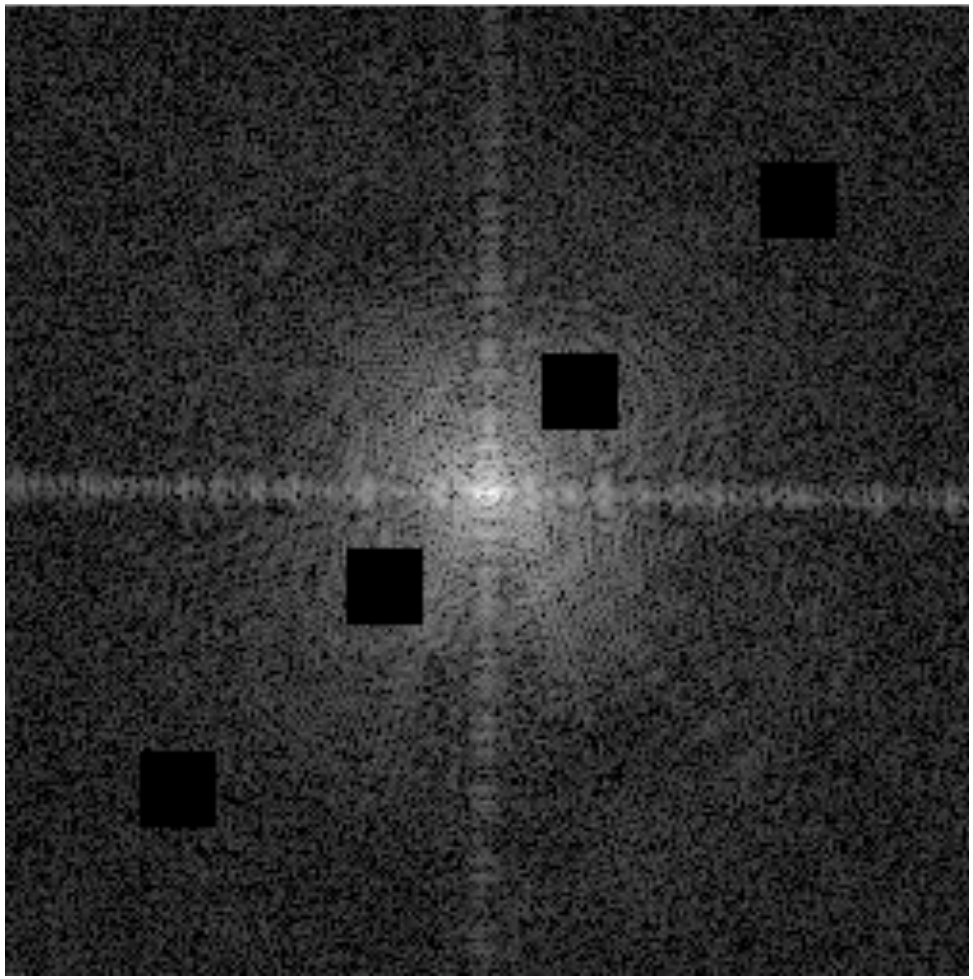
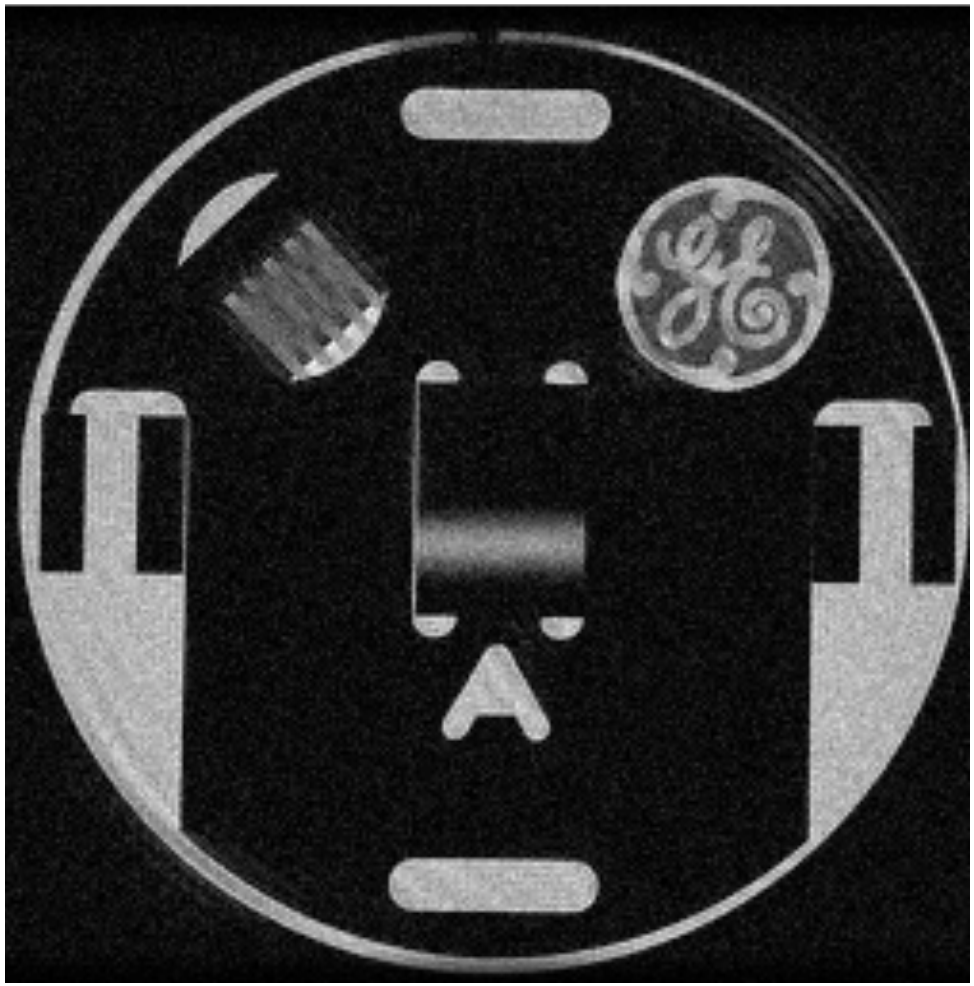
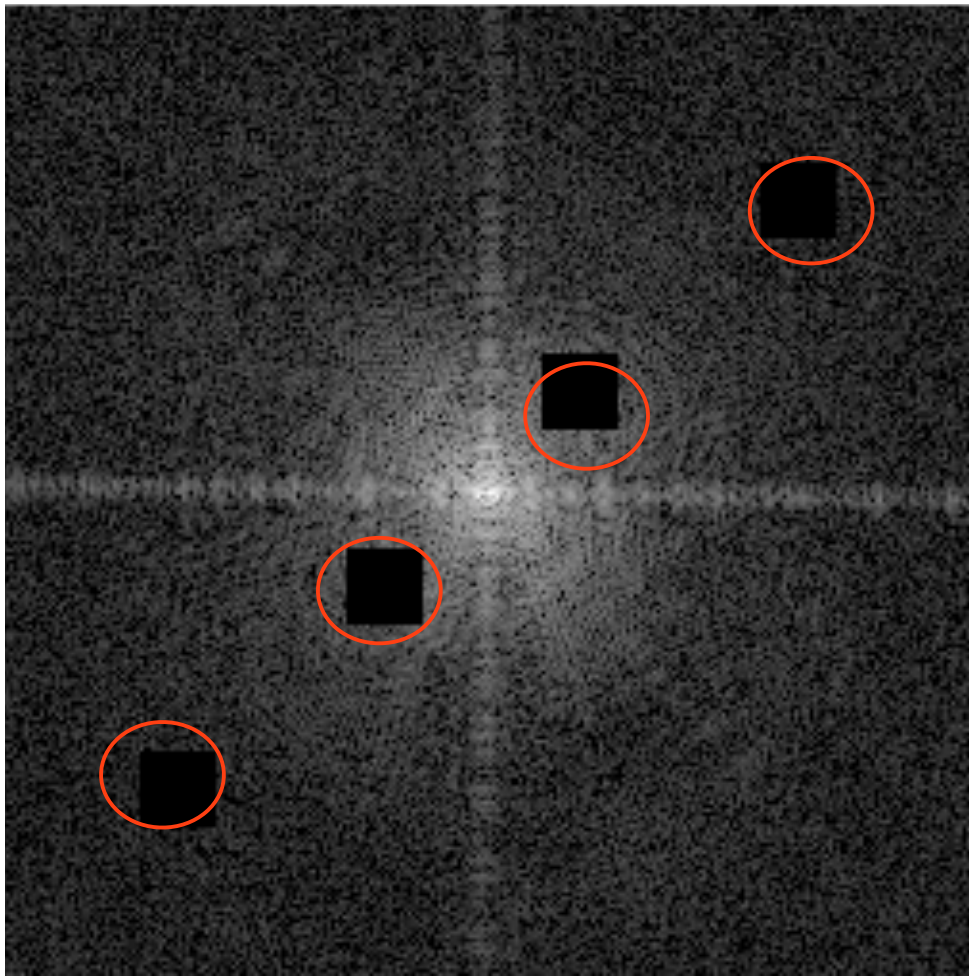
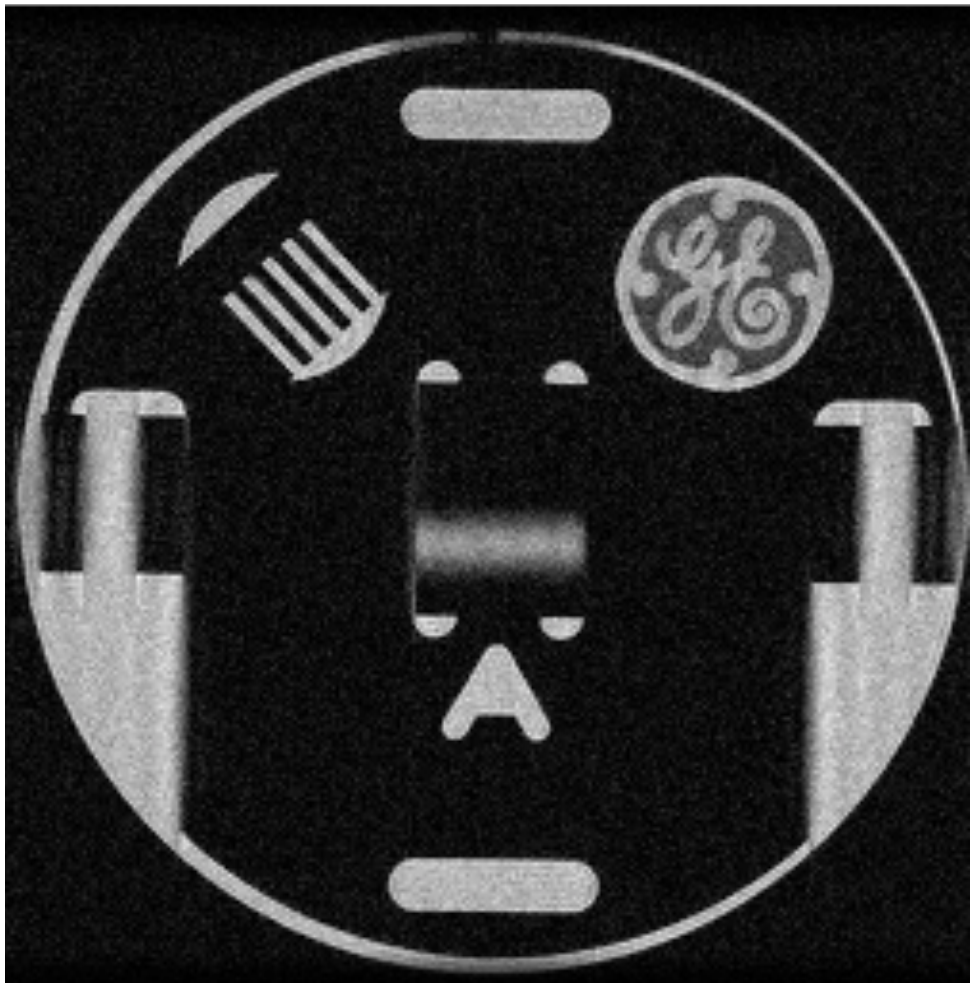
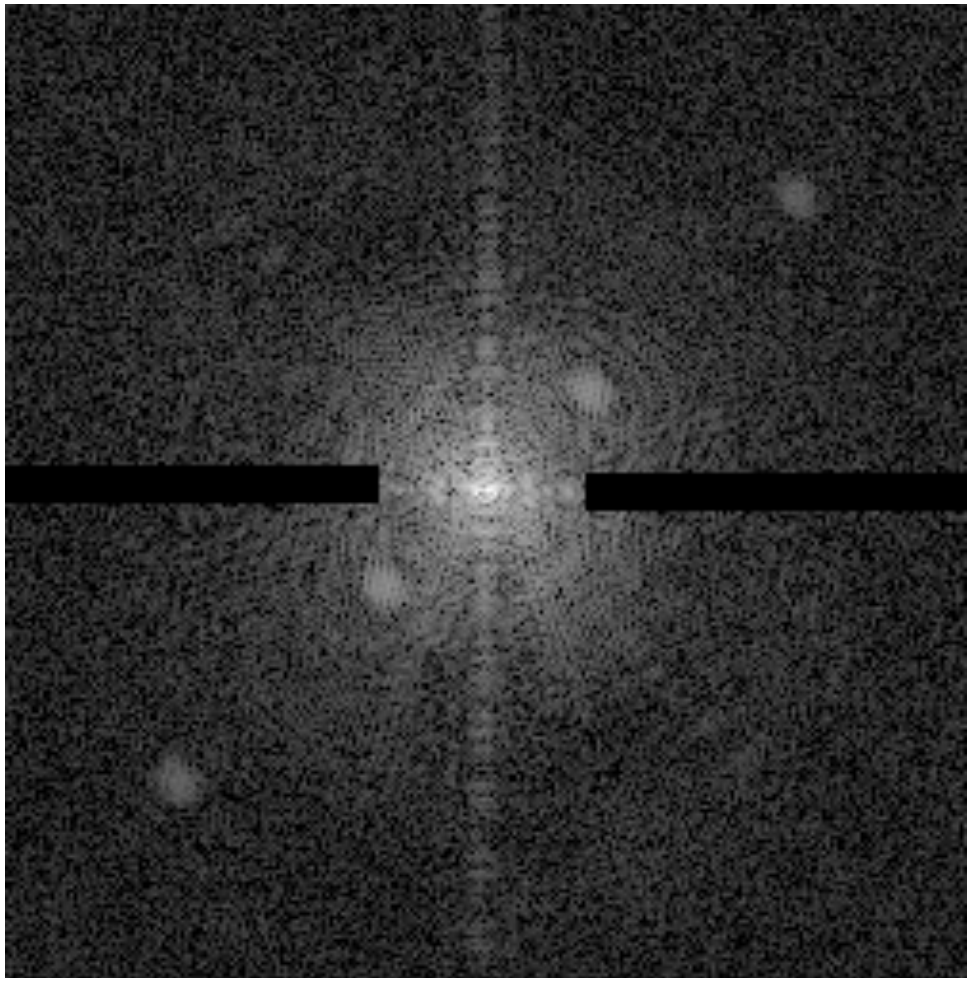
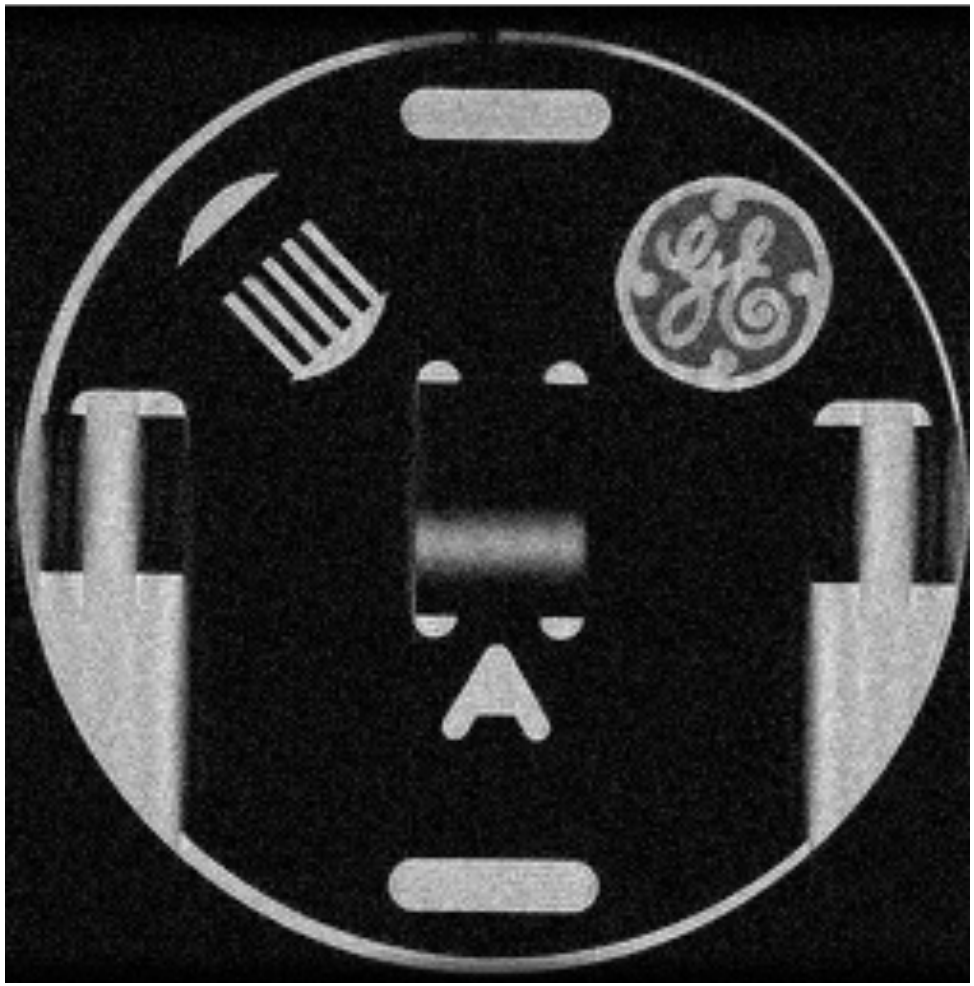
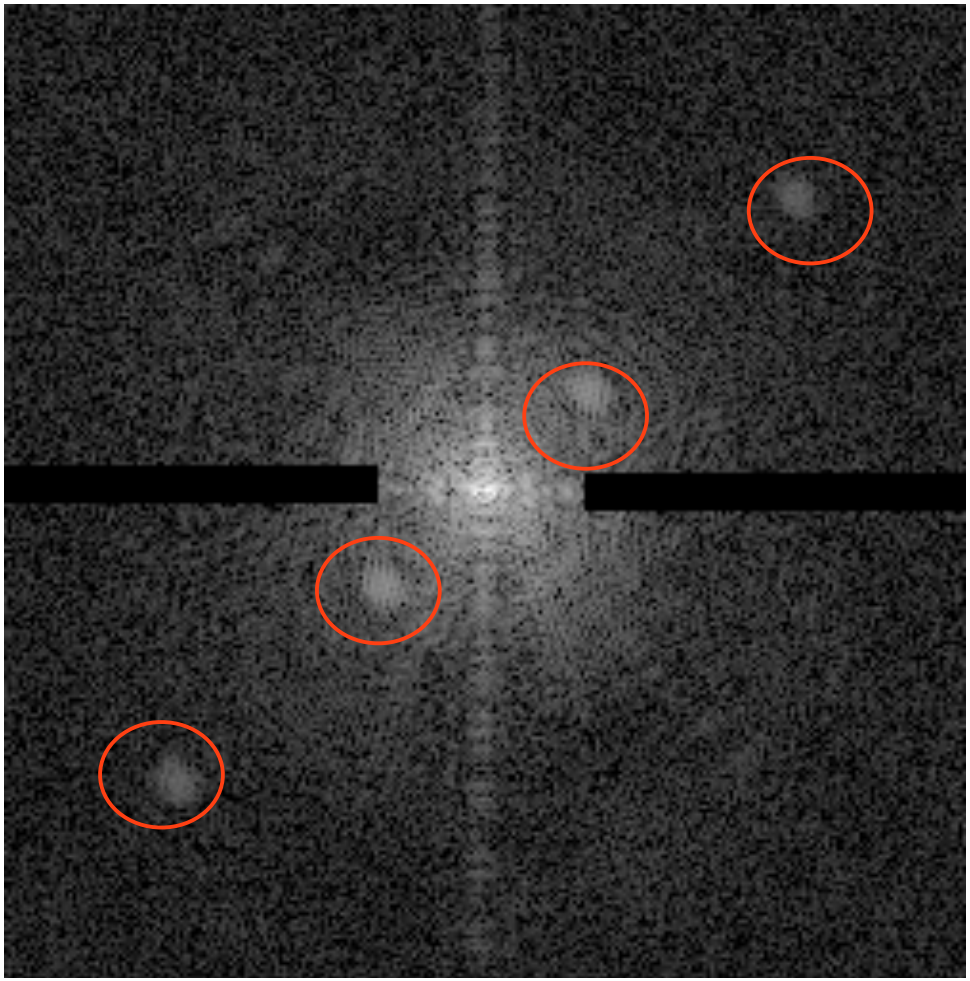
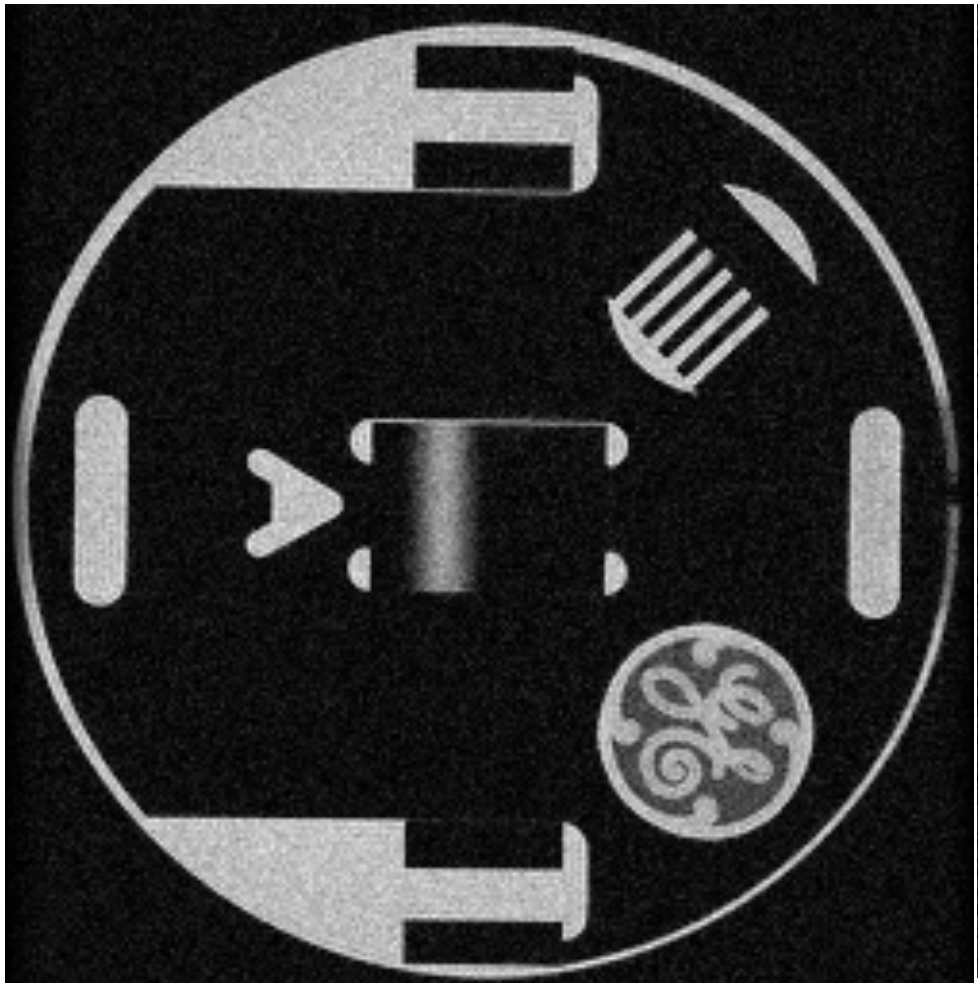
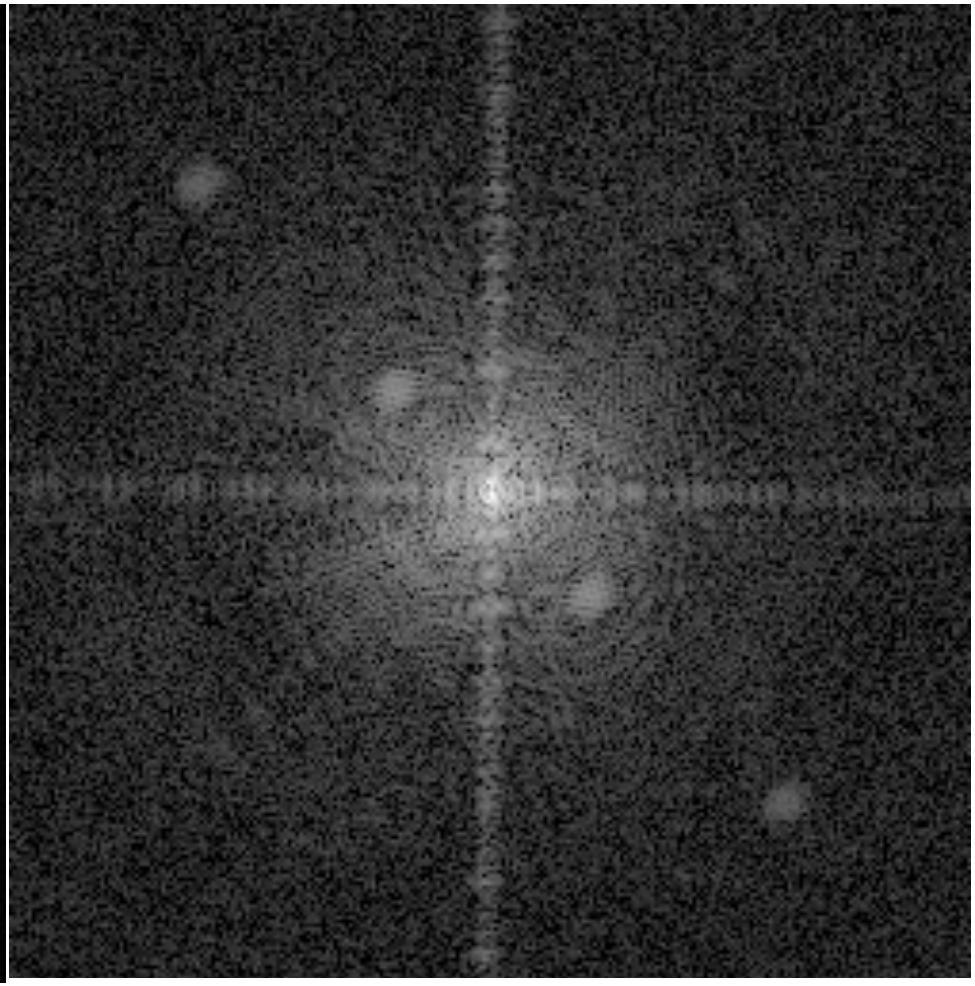
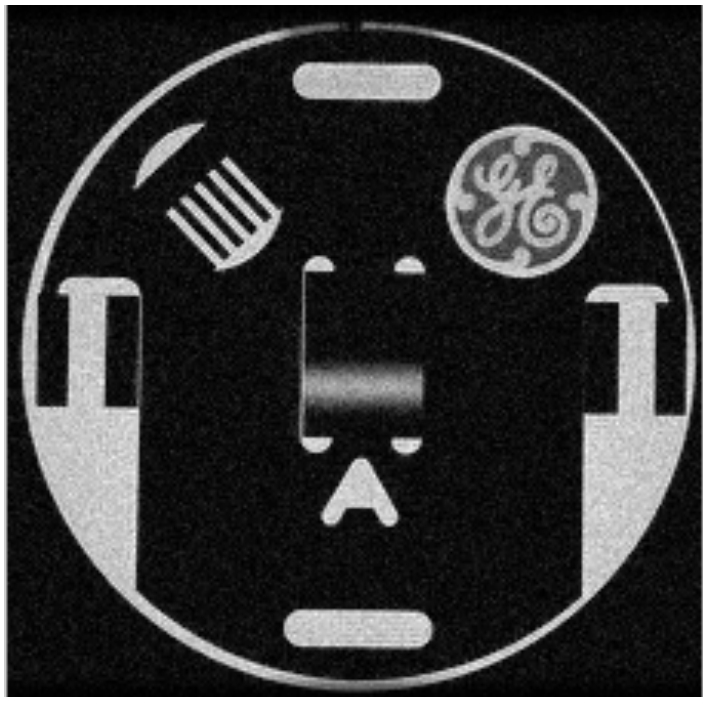
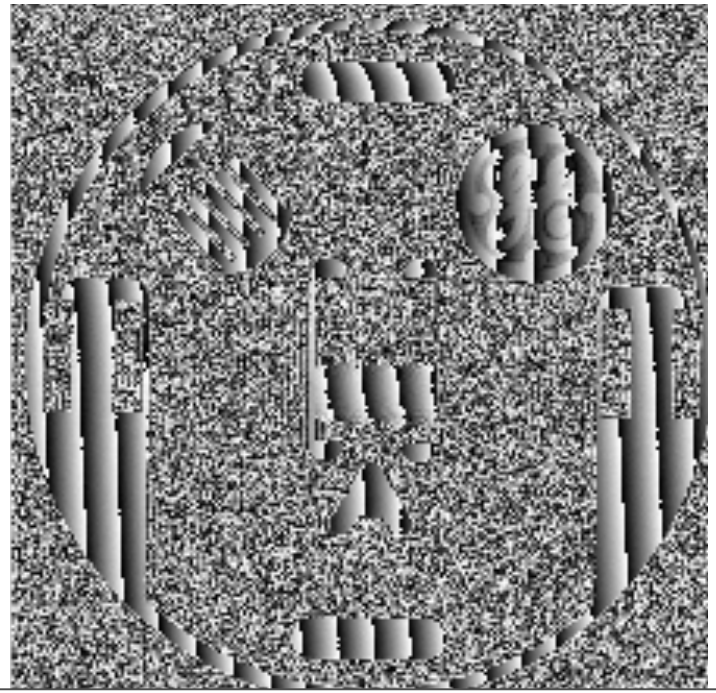
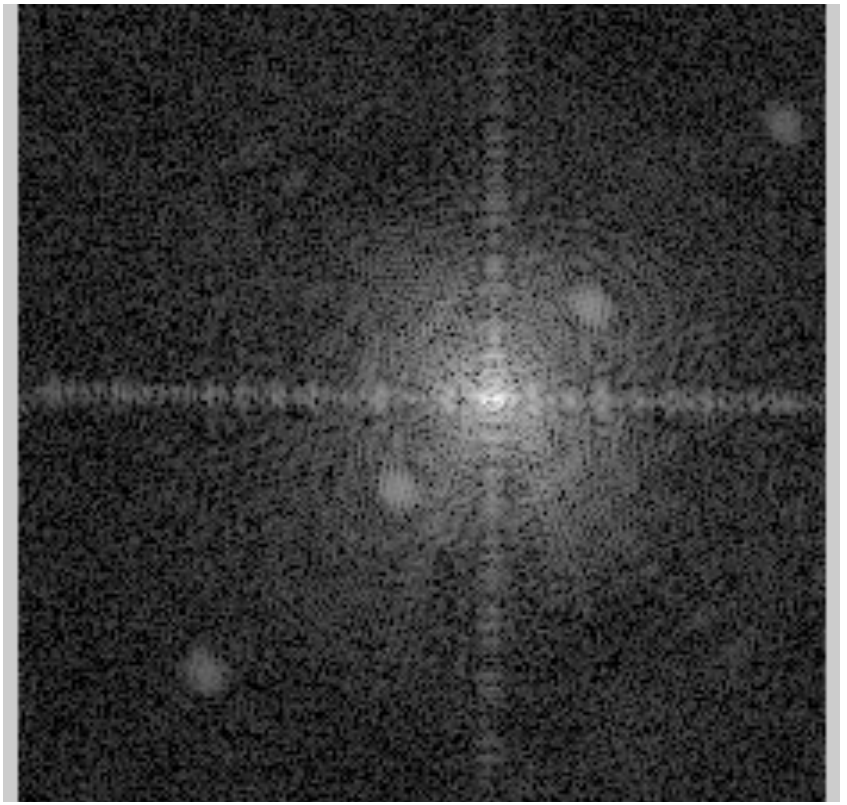






$$\frac{J_1(\pi r)}{2r}$$




Magnitude
Phase