Poisson Distribution

Poisson distribution:
a given event occurring in any time interval is distributed uniformly over the interval.

The probability of $n$ electrons arriving in a period $T$: is

$$p(n) = \frac{\bar{n}^n e^{-\bar{n}}}{n!}$$

where $\bar{n}$ is the average number of electrons arriving in $T$

Properties of Poisson Distribution:
Mean = $\bar{n}$
Variance = $\bar{n}$
Spectral Density Function

Random variable \( i(t) \) consists of a large number of individual events (e.g., single-electron photocurrent) at random time:

\[
i(t) = \sum_{j=1}^{N_T} f(t - t_j), \quad 0 \leq t \leq T
\]

Fourier transform: \( I_T(\omega) = \sum_{i=1}^{N_T} F_i(\omega) \)

\[
F_i(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t - t_i)e^{-i\omega t} \, dt = \frac{e^{-i\omega t_i}}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} \, dt = e^{-i\omega t_i} F(\omega)
\]

\[
|I_T(\omega)|^2 = |F(\omega)|^2 \sum_{i=1}^{N_T} \sum_{j=1}^{N_T} e^{-i\omega(t_i - t_j)} = N_T |F(\omega)|^2 = N_T |F(\omega)|^2
\]

\( \bar{N} \): average rate of electron arrival

Spectral density function:

\[
S(\nu) = \lim_{T \to \infty} \frac{8\pi^2 |I_T(2\pi \nu)|^2}{T} = 8\pi^2 \bar{N} |F(2\pi \nu)|^2
\]

Shot Noise

Shot Noise: Noise current arising from random generation and flow of mobile charge carriers.

Current pulse due to a single electron moving at \( v(t) \):

\[
i_e(t) = \frac{ev(t)}{d}
\]

Fourier transform: \( F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(t)e^{-i\omega t} \, dt \)

\( t_a \) : arrival time, \( x(0) = 0, \quad x(t_a) = d, \)

Small transit time \( t_a \), \( \omega t_a \ll 1 \quad \rightarrow \quad e^{-i\omega t} \sim 1 \)

\[
F(\omega) = \frac{1}{2\pi} \int_{0}^{t_a} \frac{dx}{dt} \cdot 1 \cdot dt = \frac{1}{2\pi} \int_{0}^{d} \frac{dx}{dt} = \frac{e}{2\pi}
\]

\[
S(\nu) = 8\pi^2 \bar{N} \left( \frac{e}{2\pi} \right)^2 = 2e\bar{I}
\]

\[
\bar{I} = e\bar{N}
\]

\[
i_e^2(\nu) = S(\nu)dv = 2e\bar{I}dv
\]
Thermal Noise (Johnson Noise)

- Fluctuation in the voltage across a dissipative circuit element (resistor)
- Caused by thermal motion of charged carriers

Thermal Noise Derivation

Consider two resistors connected by a lossless transmission line of length $L$:

- Voltage wave: $v(t) = A \cos(\omega t \pm k z)$
- Assume periodic condition: $kL = 2m\pi$

Mode density: $\rho(\nu) = \frac{L}{c}$

Power flow: $P = \frac{\text{Energy}}{\text{Transit Time}}$

$$P = \frac{1}{L/c} \left( \frac{L}{c} \Delta \nu \right) \left( \frac{h\nu}{e^{h\nu/k_B T} - 1} - 1 \right) = \frac{h\nu \Delta \nu}{e^{h\nu/k_B T} - 1}$$

$\frac{h\nu}{k_B T} \ll 1$

$$P = k_B T \Delta \nu = \left( \frac{\nu^2}{V_N} \left( \frac{R}{R + R} \right)^2 \right) \frac{1}{R} = \left( \frac{\nu^2}{I_N} \left( \frac{R}{R + R} \right)^2 \right) R$$

Equivalent mean square noise voltage:

$$V_N^2 = 4k_B T R \Delta \nu$$

Equivalent mean square noise current:

$$I_N^2 = \frac{4k_B T \Delta \nu}{R}$$
Noises in p-i-n photodiodes: shot noise and thermal noise

\[ i_N^2(v) = i_{N,\text{shot}}^2(v) + i_{N,\text{thermal}}^2(v) = 2e\bar{I}dv + \frac{4k_BT\Delta v}{R} \]

Signal:

\[ i_S^2(v) = \bar{I}^2 \]

Signal to noise ratio (SNR):

\[ SNR = \frac{\bar{I}^2}{2e\bar{I}dv + \frac{4k_BT\Delta v}{R}} \]

Note that the SNR improves with increasing average photocurrent \( \bar{I} \).