Interaction of matter and radiation

[Reading assignment, Suelto, Ch. 2]

- Thermal radiation. Thermal equilibrium between two masses can be achieved by several processes.

**Conduction**

\[
\begin{array}{c|c|c}
M_1 & T_1 & \text{M}_1T_1 \\
\end{array}
\]

**Convection**

\[
\begin{array}{c|c|c|c}
M_1 & T_1 & \text{gas} & T_2 \\
\end{array}
\]

gas transports energy to reach equilibrium

**Radiation**

\[
\begin{array}{c|c|c|c}
M_1, T_1 & \text{vacuum} & \text{M}_2, T_2 \\
\end{array}
\]

\(T\) to reach equilibrium, energy must radiate, energy & absorb energy.

Hot bodies radiate light. Think of glowing coals. Boltzmann's law says that for two energy levels of a system in thermal equilibrium, the population ratio must be:

\[
\frac{N_2(E)\,dE}{N_1(E)\,dE} = \frac{g_2(E)}{g_1(E)} \frac{e^{-(E_2 - E)/kT}}{e^{-(E_1 - E)/kT}}
\]

for a continuous spectrum of energy levels (i.e. a solid)

where here, \(g(E)\) is the density of states function.
To get visible light (~2 eV photons), consider Boltzmann's ratio.

\[ a \times T = 300 \text{ K}, \ K_T = 1.026 \text{ eV} \]
\[ e^{-2/1.026} \approx 4 \times 10^{-34} \text{ very small population!} \]

\[ a \times T = 1000 \text{ K}, \ K_T = 0.87 \text{ eV} \]
\[ e^{-2/0.87} \approx \]

\[ a \times T = 5000 \text{ K}, \ K_T = 0.43 \text{ eV} \]
\[ e^{-2/0.43} \approx \]

Thermal radiation - total radiation increases sharply with temperature, and peak shifts to shorter with \( T \).

Stefan-Boltzmann law:
- total intensity radiated by surface of a hot body.

Stefan-Boltzmann constant \( \sigma = 5.67 \times 10^{-8} \text{ W/cm}^2 \cdot \text{K}^4 \)

Emissivity \( \varepsilon_m \) - dimensionless \( 0 \leq \varepsilon_m \leq 1 \)
- ability of body to emit + absorb energy.

If \( M_1 \) is initially at higher \( T \), it radiates more and heats up \( M_2 \). But eventually, \( M_2 \) must radiate as much as it absorbs to reach equilibrium.
Power/area incident on \( H_2 \) : \( I \) in "irradiance"

Radiated by \( H_2 \) : \( H_2 \) in "radiance" 

\[ I \] is fraction of incident power that is absorbed

If several meshes are put inside the cavity, the intensity would be the same for all, so

\[ I = \frac{I_1}{b_2} = \frac{I_3}{b_3} = \cdots \]

This says the ratio of power radiated to fraction absorbed is a constant independent of material.

Since \( H_2 = E m_2 + T^4 \), and in equilibrium all meshes reach the same \( T \), we can find \( E m = 0 \)

Perfect reflector \( \rightarrow E m = 0 \)

Lamp black \( \rightarrow E m = 0.95 \)

For the cavity inside, \( I = 0 \) \( T^4 \). If we open a small hole and allow \( I \) to come out, we have a perfect black body radiator at the hole.

Planck theory of cavity radiation

Calculate the spectral energy density for radiation in thermal equilibrium with walls of cavity.

Total energy density in the e-m field: \( \rho \)

Spectral energy density: \( \rho \nu \)

\[ \rho = \int \rho \nu \ d\nu \]

Spectral Intensity

(Note: for a beam travelling in a specific direction, \( I = \frac{\rho \nu}{m} \).

But integrating light coming out of the hole over \( 2\pi \) solid angle, taking into account \( \cos \theta \) factor for obliquity, we get the factor of \( 4 \).)
For cavity radiation, $P_v$ is calculated as product of

$\left[ \text{# of modes/unit volume} \times \text{unit frequency} \right] \times \left[ \text{average energy/mode} \right]$

Similar to density of states

Mode counting: solve wave equation in a rectangular box

$E(x, y, z) \sim e^{i(k_x x + k_y y + k_z z)}$

$k^2 = k_x^2 + k_y^2 + k_z^2$

$k = \frac{2\pi}{\lambda} = \frac{2\pi \nu}{c}$

$n_x, n_y, n_z \; : \text{integer}$

Total # of modes with frequency from $0 \to \nu$ is

Volume of one octant of sphere in $k$-space

$N(\nu) = \frac{\frac{1}{8} \times \frac{4}{3} \pi (\frac{2\pi \nu}{c})^3}{(\pi/2x)(\pi/2y)(\pi/2z)} \times 2 = \frac{8\pi \nu^3}{3c^3}$

2 polarizations per mode

We want # modes/freq-vol:

Average energy/mode = $\frac{\hbar \nu}{\text{occupation probability}}$

for photons: Bose-Einstein distribution

$\langle E \rangle = \frac{\hbar \nu}{e^{\hbar \nu/kT} - 1}$

So,

$P_v = \frac{8\pi \nu^2}{c^3} \frac{\hbar \nu}{e^{\hbar \nu/kT} - 1}$

$I_v = \frac{2\pi \hbar \nu^3}{c^2} \frac{1}{e^{\hbar \nu/kT} - 1}$

($\# W: \text{derive S-B law, find } T$)
Atomic radiative transitions
- Classical electron oscillator model

Radiation from transitions occurs at discrete frequencies. Atom appears to behave as a simple oscillator model - electron on a spring

$$\text{spring constant, } K$$

Assuming no damping, simple spring eqn:

$$m_e \frac{d^2 x}{dt^2} + Kx = 0$$

$$x = x_0 e^{-j\omega t}$$

Oscillating electron radiates. Total power radiated by a dipole is

$$P = \frac{dE}{dt} = -Y_0 E$$

$E$: total energy in the spring system

$$E = E_0 e^{-\gamma_0 t} = E_0 e^{-t/\tau_0}$$

Radiative damping - add term $Y_0 \frac{dx}{dt}$ to eqn of motion

$$m \frac{d^2 x}{dt^2} + Y_0 \frac{dx}{dt} + m_0 \omega_0^2 x = 0$$

$$x = x_0 e^{-\gamma_0 t}$$

Electric field radiated has same time dependence:

$$I(\theta) = |E(\theta)|^2 = I_0 e^{-\gamma_0 t}$$
Emission spectrum of radiating electron

\[ = \frac{E_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(\omega - \omega_0) + j \gamma_0/2} dt \]

\[ = -\frac{E_0}{\sqrt{2\pi}} \frac{1}{\sqrt{[(\omega - \omega_0) + j \gamma_0/2]^2}} \]

Intensity spectrum \( I(\omega) = |E(\omega)|^2 \)

\[ I(\omega) = I_0 \frac{\gamma_0/2\pi}{[(\omega - \omega_0)^2 + \gamma_0^2/4]} \]

\[ \text{Total intensity} \]

\[ I_0 = \int_0^\infty I(\omega) d\omega \]

"Lorentzian lineshape"

Quantum mechanical treatment

- Rigorous QM treatment is beyond our scope. Need QM theory for radiating field.
- Semiclassical treatment. QM for atoms, classical for field.

Radiation occurs during transition from level 2 to level 1. The atom develops a quantum mechanical dipole moment that radiates.
\[ y_1(r, t) = u_1(r) e^{-j \frac{E_1}{\hbar} t} \quad y_2 = u_2(r) e^{-j \frac{E_2}{\hbar} t} \]

\( u_1, u_2 \) are stationary eigenfunctions of the atom

During the transition, \( y = a_1(t) y_1 + a_2(t) y_2 \)

Electric dipole moment of atom

\[
= -e \int \mathbf{r} |a_1|^2 |u_1|^2 \, dv - e \int \mathbf{r} |a_2|^2 |u_2|^2 \, dv \\
- e \int \mathbf{r} [a_1 a_2^* u_1 u_2^* e^{j \omega t} + c.c.] \, dv \quad \omega_0 = \frac{E_2 - E_1}{\hbar}
\]

Oscillating dipole \( \mu_{osc} = \text{Re} \left[ 2a_1 a_2^* u_1 \right] e^{j \omega_0 t} \)

\( \mu_{21} = \int a_2^* \mathbf{r} u_1 \, dv \) \quad electric dipole matrix element

If we write the QM power radiated per atom:

\[ A_{21}: \text{spontaneous emission rate} \]

Compare to classical expression:

\[ P_2 = \frac{\mu^2 \omega_0^4}{12 \pi \epsilon_0 c^3} \]

Identify \( \mu \) with \( 2 \mu_{21} \): dipole matrix element, then

\[ A_{21} \hbar \omega_{21} = \frac{4 \mu_{21}^2 \epsilon_0 c^4}{12 \pi \epsilon_0 \hbar c^3} \]

or

\[ A_{21} = \frac{\mu_{21}^2 \omega_{21}^3}{3 \pi \epsilon_0 \hbar c^3} \approx \frac{1}{\kappa_{spn}} \]