Absorption and Stimulated Emission

Interaction Hamiltonian for electron in radiation field $E = E_0 \sin \omega t$

dipole approximation

for absorption of light, transition from 1 to 2

Fermi's golden rule says

$W_{12} = \frac{\hbar^2}{3\hbar^2} 1 \langle 2 | \vec{\mu} | 1 \rangle^2 E_0^2 \delta (\nu - \nu_0)$

where $\langle 2 | \vec{\mu} | 1 \rangle = -e \int \frac{dV \cdot \hat{r}}{\hbar^2} \delta (\rho - \rho_0)$

Since there is always some broadening of energy levels

$W_{12}^a = \frac{\hbar^2}{3\hbar^2} | \langle 2 | \vec{\mu} | 1 \rangle |^2 E_0^2 g(\nu - \nu_0)$

where $g(\nu - \nu_0)$ is normalized lineshape $\int g(\nu - \nu_0) d\nu = 1$

e.g. $g(\nu - \nu_0) = \frac{1}{\pi \Delta \nu_0} \frac{1}{1 + [2(\nu - \nu_0)/\Delta \nu_0]^2}$


Using

we can write $| E_0 |^2 = \frac{2f}{\hbar^2 \epsilon_0} = \frac{2}{\hbar^2 \epsilon_0} c$

$W_{12}^a = \frac{2\pi^2}{3\hbar^2 \epsilon_0 c} | \langle 2 | \vec{\mu} | 1 \rangle |^2 g(\nu - \nu_0)$

Calculation is exactly the same for the stimulated $2 \rightarrow 1$

transistor

$W_{21}^{st} = W_{12}^a$
Einstein treatment

1. Put atoms in thermal equilibrium with radiation field
2. Thermal equilibrium for field is Planck distribution
3. Thermal equilibrium for atoms is Boltzmann distribution

Recall

\[
\frac{N_2}{N_1} = \frac{(E_2 - E_1)}{kT} \quad \text{Boltzmann distribution}
\]

Let \( W_{12}^a = B_{12} f_{x_0} \quad W_{21}^s = B_{21} f_{x_0} \)

In equilibrium, steady state, total 1 \( \rightarrow \) 2 rate must equal the total 2 \( \rightarrow \) 1 rate.

\[
W_{12}^a N_1 = W_{21}^s N_2 + A_{21} N_2
\]

\[
B_{12} f_{x_0} N_1 = B_{21} f_{x_0} N_2 + A_{21} N_2
\]

\[
\frac{B_{12} f_{x_0} N_1}{B_{21} f_{x_0} N_2} = \frac{N_2}{N_1} (B_{21} f_{x_0} + A_{21}) = e^{\Delta H / kT} (B_{21} f_{x_0} + A_{21})
\]

Solving for \( f_{x_0} \), we find

\[
\text{Compare to known formula for } f_{x_0} \text{ (Planck), we require}
\]

\[
B_{12} = B_{21} \quad \frac{A}{B} = \frac{8\pi^2 h^2/\hbar^3}{c^3}
\]

Since \( W_{12} \) can be written \( W_{12} = \frac{2\pi^2}{3\hbar c^2} W_{12}^a f_{x_0} \), we find

\[
B = \frac{2\pi^2/\hbar c^2}{3\epsilon_0 h^2} \quad \text{and}
\]

\[
A = \frac{16\pi^3 h^2/\hbar c^2}{3\epsilon_0 c^2}
\]
Selection rules

The transition probabilities are all proportional to $|M_{12}|^2$

where

atomic wave functions have definite parity,

1. $\psi(-r) = \psi(r)$ even parity
2. $\psi(-r) = -\psi(r)$ odd parity

So, for $\ell_1 \ell_2$ to have non-zero value, $\ell_1$ and $\ell_2$ must have opposite parity. This is a selection rule. Certain transitions are forbidden by selection rules.

Non-radiative decay

- $\ell_2^2$ Level 2 can decay to level 1 by other mechanisms besides spontaneous radiative emission.

Collisional decay. In a gas, excited atoms can have collisions with other atoms. The internal energy can be converted to kinetic energy, shared among the two atoms. It can also be taken up by exciting the other atom to an excited state (if energy can be conserved). Denote excited atom as $A^*$. 

1st process

2nd process

\[ \ell_2^2 \rightarrow \ell_2 \ell_3 \]

atom A \[ \ell_2 \ell_3 \]
Line broadening

We previously showed that a classical oscillator with radiative damping, such that

$$I(\omega) = I_0 \frac{\gamma_0/4\pi}{[\omega - \omega_0]^2 + \gamma_0^2/4} \quad \gamma_0 = 1/\tau_0$$

where $E$ is the stored energy in the oscillator, has an emission spectrum that is Lorentzian.

Quantum mechanically, the population density in level 2 decays exponentially. A detailed quantum treatment of spontaneous emission shows that the power emitted by a single atom decays as $\exp(-t/\tau_0)$. We can also model the radiated electric field as

Thus the power spectrum of spontaneous emission is Lorentzian. It is also true, using thermodynamic arguments, that the absorption and stimulated emission spectra must be the same as the spontaneous emission spectrum.

This broadening mechanism gives the “natural linewidth” or “spontaneous emission linewidth”.

“Collision broadening.” Atoms or molecules in a gas undergo collisions. If a collision occurs during a radiative transition event, it can interrupt the phase of the radiation (dephasing collision).

\[ \text{\includegraphics[width=0.5\textwidth]{wiggly}} \]
Kinetic theory of gases tells us that collisions in a gas occur with an average interval $\tau_c$, which depends on gas density and temperature. The time between collisions obeys a probability distribution:

$$
\Gamma(\tau) \equiv \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E^*(t) E(t+\tau) dt
$$

$$
= E_0^2 \exp\left[j\omega \tau - (\chi_0 \tau^2 / 2c)\right]
$$

The power spectrum can be obtained from the Fourier Transform of $\Gamma(\tau)$:

$$
g(\omega - \omega_0) = \frac{2c}{\pi} \frac{1}{[1 + (\omega - \omega_0)^2 c^2]} \quad \text{normalized}
$$

or

$$
g(\nu - \nu_0) = 2c \frac{1}{\pi c^2} \frac{1}{[1 + 4\pi^2 c^2 (\nu - \nu_0)^2]^{1/2}}
$$

For radiative broadening, the linewidth (FWHM)

$$\Delta \nu_p = \frac{1}{2\pi \tau_p}$$

For collision broadening, the linewidth is

$$\Delta \nu_c = \frac{1}{\pi c^2}
$$

When both are present, $\Delta \nu_{tot} = \Delta \nu_p + \Delta \nu_c$.

From kinetic theory,

$$
\nu_b = \frac{1}{\pi \rho c^2}
$$

Where $\rho$ is atomic mass, $P$ is pressure, $T$ is temperature, $c$ is atomic hard sphere radius.
The collisional linewidth is proportional to pressure, so collision broadening is also known as pressure broadening.

**Inhomogeneous broadening**

Another broadening mechanism is when we have a collection of atoms with some distribution of resonant frequencies. For atoms in a gas there is the Doppler effect.

For light waves travelling in the direction and atom velocity component along z, \( v_z \), the atom sees a shifted frequency:

\[
\nu' = \frac{\nu_0}{\sqrt{1 - v_z^2/c^2}} = \nu_0 \left( 1 + \frac{v_z}{c} \right)
\]

If the atomic resonance is \( \nu_0 \), then for the moving atom, the apparent resonance in the lab frame is

\[
\nu_0' = \frac{\nu_0}{\sqrt{1 - v_z^2/c^2}} = \nu_0 \left( 1 + \frac{v_z}{c} \right)
\]

In a gas atom velocities are given by Maxwell distribution. The probability that an atom has a velocity component \( v_z \) is

\[
\frac{d\nu_0}{\nu_0} = \frac{d\nu_0}{\sqrt{2\pi k T}} e^{-\frac{1}{2kT} \left( \frac{\nu_0 - \nu_0'}{\nu_0} \right)^2}
\]

Using \( v_z = c \left( \frac{\nu_0'}{\nu_0} - 1 \right) = c (\nu' - \nu_0)/\nu_0 \), and

\[
\frac{d\nu_0}{\nu_0} = \frac{d\nu_0}{\sqrt{2\pi k T}} e^{-\frac{1}{2kT} \left( \frac{\nu_0 - \nu_0'}{\nu_0} \right)^2}
\]

we get

\[
\frac{d\nu_0}{\nu_0} = \frac{1}{\nu_0} \left( \frac{Mc^2}{2\pi k T} \right)^{1/2} e^{-\frac{1}{2kT} \left( \frac{\nu_0 - \nu_0'}{\nu_0} \right)^2}
\]

This assumes each atom has a Gaussian spectrum. The FWHM linewidth is

\[
\Delta \nu_0^* = 2 \nu_0 \left( \frac{2kT m c^2}{1} \right)^{1/2}
\]
The "Doppler lineshape" has a Gaussian shape instead of Lorentzian. If the individual atomic lineshape (homogeneous lineshape) is Lorentzian, then the net lineshape is a convolution.

\[ \text{Doppler lineshape} \]

Individual velocity group with Lorentzian line shape

The total lineshape function is:

\[ g^{\text{inh}} \text{ is the inhomogeneous lineshape, and} \]
\[ g^{\text{hom}} \text{ is the homogeneous lineshape.} \]

The transition rate and cross-section are proportional to \( g \)

\[ W = \frac{8 \pi^2}{3 \hbar c} \mu^2 g \delta(v - v_0) I \]
\[ \Gamma = \frac{2 \pi^2}{3 \hbar c} \nu g \delta(v - v_0) \]

Recall absorption coefficient.
Saturation of absorption

\[ \frac{dN_2}{dt} = -W(N_2 - N_1) - \frac{N_2}{\tau} \]

let \( N_{\text{tot}} = N_1 + N_2 \)

\[ \Delta N = N_1 - N_2 \]

so \( N_2 = \frac{1}{2}(N_{\text{tot}} - \Delta N) \)

or \( \frac{d\Delta N}{dt} = -\Delta N(2W + \frac{1}{\tau}) + \frac{N_2}{\tau} \)

at steady state, \( \frac{d\Delta N}{dt} = 0 \), so

\[ \Delta N = \frac{N_2/\tau}{2W + \frac{1}{\tau}} = \frac{N_2}{1 + 2W \tau} \]

since \( W = \frac{\alpha \tau}{h \nu} \)

\[ I_{\text{sat}} = \frac{h \nu}{2 \tau \alpha} \]

For \( I > I_{\text{sat}} \), \( \Delta N \to 0 \). We can write the absorption coeff. as

\[ \alpha = \frac{\alpha_0}{1 + I/I_{\text{sat}}} \]

\[ \alpha_0 = \frac{2\pi^2}{3} \sin \gamma N \delta g(V - V_0) \quad (\text{unsaturated absorption coefficient}) \]

Notice that the absorption spectrum is scaled by \( \frac{1}{1 + I/I_{\text{sat}}} \)

lineshape remains the same
probe experiment:

\[ I(\nu) \]

strong beam

weak probe \( I'(\nu) \)

\[ I/\text{Isat} = 0 \]

\[ I/\text{Isat} = 1 \]

\[ I/\text{Isat} = 5 \]

\[ \nu_0 \]

Gain saturation

\[ \frac{N_2}{N_1} \]

\[ \frac{\frac{N_2}{E_2}}{\frac{N_1}{E_1}} \]

To produce an inversion on the \( 2 \rightarrow 1 \) transition, we need a supply of population in level 2, not coming directly from level 1. Rate is \( R_p \).

Also, level 1 decays rapidly to lower levels. Assume \( \tau_1 < \tau_2 \). Then

In steady state \( \frac{\partial N_2}{\partial t} = 0 \)

\[ N_2 = \frac{R_p \tau_2}{1 + \text{Isat} \tau_2} \]

rewrite: \[ N_2 = \frac{R_p \tau_2}{1 + I/\text{Isat}} \]

\[ \text{Isat} = \frac{hv}{\Delta E_2} \]

The gain coefficient is proportional to \( N_2 \), since \( N_1 \leq 0 \)

so

\[ g_0 = \sigma R_p \tau_2 \]

For homogeneous broadening, gain saturates in similar fashion to absorption.
Inhomogeneous saturation

The total line shape is a combination of narrow homogeneous lines at various center frequencies. For a strong saturating beam at $y_0$, only atoms with center frequency near $y_0$ are induced to make transitions. So the absorption (or gain) is only saturated for these atoms. A probe beam will experience the spectrum shown in $B$ with a "hole" in the vicinity of $y_0$. The width of the hole is given by the homogeneous linewidth.