Optical Resonators

Suppose we have a certain Gaussian beam

\[ M_{1,1}, \quad z = 0 \quad \text{waist} \quad M_{2,2} \]

Place mirror \( M_{1,2} \) at \( z_{1,2} \) with radius \( R_{1,2} \) to match Gaussian beam radii.

Since mirror surfaces are normal to energy propagation direction, the beam reflects perfectly back on itself.

Beam is trapped. "Reproducing stable field" configuration results.

(Mirrors must be large enough so spillover past them is negligible.)

It follows that if mirror is partially transmitting, a Gaussian beam emerges.

(Mirror might act as a lens to transform output beam)
How about if we choose mirror radii and positions: Can we find the Gaussian beam that fits inside?

If we can find $z_1, z_2$, and $w_0(z_0)$ (relative to $z_0$ - position of beam waist), we know everything:

$$ R_1(z_1) = z_1 + \frac{z_1^2}{R_1} = -R_1 $$

$$ R_2(z_2) = z_2 + \frac{z_2^2}{R_2} = R_2 $$

Solve for $z_1, z_2, z_R$

Define $g_1 = 1 - \frac{L}{R_1}$, $g_2 = 1 - \frac{L}{R_2}$ "resonator $g$-parameters"

Solutions are:

$$ z_2^2 = \frac{g_1 g_2 (1-g_1 g_2)}{(g_1+g_2-2g_1 g_2)^2} L^2 $$

$$ z_1 = \frac{g_2 (1-g_1)}{g_1+g_2-2g_1 g_2} L $$

You check!
also \( \omega^2_0 = \frac{\lambda Z_0}{\pi} \)

\[ = \frac{L}{\pi} \sqrt{\frac{g_2 (1-g_2 g_1)}{(g_1 + g_2 - 2g_2 g_1)^2}} \]

\[ \omega^2_1 = \frac{L}{\pi} \sqrt{\frac{g_2}{g_1 (1-g_2 g_1)}} \quad \omega^2_2 = \frac{L}{\pi} \sqrt{\frac{g_1}{g_2 (1-g_2 g_1)}} \]

for \( \omega_0, \omega_1 \) and \( \omega_2 \) to be real and finite

clearly

Resonator stability diagram:

[Diagram showing the stability region with \( g_2 = 1 - \frac{4}{\pi} \) and \( g_1 g_2 = 1 \) as critical points.]
Example: Symmetric resonator \( R_1 = R_2 \)

\[
\omega_1^2 = \omega_2^2 = \frac{\pi \lambda}{2B} \sqrt{1 - g^2}
\]

\( \omega_1, \omega_2 \)

\( R = \frac{L}{2} \)

Confocal resonator

Plane parallel resonator

Often desire large mode volume in laser to extract maximum energy from gain medium

Good

Bad
Numbers: assume \( \lambda = 600 \text{ nm}, \quad L = 10 \text{ cm} \).
Suppose we have a 3 mm diameter laser rod. Design for \( d \approx 3 \omega_0 \) or \( \omega_0 = 1 \text{ mm} \).

\[ \chi^2 \]

\[ 4 \chi^2 (1 - g) = 1 + g \]

\[ \frac{4 \chi^2 - 1}{4 \chi^2 + 1} = g \]

\[ \chi^2 = 2742, \quad g = 1 - 1.82 \times 10^{-4} \]

Planar resonators very unstable alignment.

Practical: \( R = 10 \text{ m} \Rightarrow \omega_0 = 0.04 \text{ cm}, \quad 1.4 \text{ mm} \)

Mirror diameter should be large to cover full beam \( d \approx 3 \omega_0 \).

To more efficiently fill laser mode volume, put additional optics inside cavity.

Example:

- Laser mirror
- Beam expanding
- Telescope
General resonator analysis

\[
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} \Rightarrow
\begin{array}{c}
\tilde{q}_2 \\
\tilde{q}_1
\end{array}
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

Model complete resonator by single $A B C D$ matrix. Include laser mirrors, any intra-cavity optics, and free space propagation \(\rightarrow\) one complete round trip.

Choose reference plane. Input Gaussian beam with parameter $\tilde{q}_1$. After one full round trip, beam comes back around with parameter $\tilde{q}_2$.

Resonator mode is self-consistent \(\tilde{q}_2 = \tilde{q}_1 = \tilde{q}\)

\[
\tilde{q} = \frac{A\tilde{q} + B}{C\tilde{q} + D}
\]

\[
\tilde{q}^2 + \frac{(A-D)}{B} \tilde{q} - \frac{C}{B} = 0
\]

\[
\tilde{q}_{1,2} = \frac{(1-D)}{2B} \pm \frac{1}{2B} \sqrt{(A-D)^2 + 4BC}
\]

Using $AD-BC = 1$

\[
\tilde{q}_{1,2} = \frac{D-A}{2B} \pm \frac{1}{2B} \sqrt{(A+D)^2 - 4}
\]
Recall \( \frac{1}{q} = \frac{1}{r} - i \frac{\Delta}{\pi w^2} \).

\( \frac{1}{q} \) must have a negative imaginary part.

So \((A+D)^2 < 4\), or

\[ V_{\text{ax}} = \frac{c}{2L} \]

Most take the root. Only one unique stable Gaussian mode.

\[ + \text{ root } \rightarrow \text{ stable mode} \]

\[ - \text{ root } \rightarrow \text{ unstable mode} \]

**Mode Frequencies**

Mode most also reproduce itself in phase. Total round trip phase shift = 2\( \pi \).

Condition sets mode frequencies.

(Recall simple analysis giving)

Go back to simple 2-mirror resonator. Phase shift from 1 mirror to the other

\[
\begin{array}{c}
\mathbb{Z}_1 \\
\downarrow \quad \uparrow \\
\mathbb{Z}_2
\end{array}
\leftrightarrow \quad \uparrow \\
\downarrow \quad \uparrow \\
\mathbb{Z}_1
\]

\( \phi(\mathbb{Z}_2 - \mathbb{Z}_1) = k \pi - (n+m+1)[\eta(\mathbb{Z}_2) - \eta(\mathbb{Z}_1)] \)

Phase shift depends on higher order mode \#s.

It is possible to show (tedious algebra),

\[
\eta(\mathbb{Z}_2) - \eta(\mathbb{Z}_1) = \cos \pm \sqrt{g_1 g_2} \quad \text{sign \rightarrow pos branch} \\
-\text{sign \rightarrow neg branch}
\]
Round trip phase shift is just twice this.

So mode frequencies determined by:

\[ V_{snm} = \frac{W_{snm}}{2\pi} = \frac{c}{2L} \left[ S + (n+m+1) \frac{a \pi}{\pi} \right] \]

as before

Mode frequencies for higher order modes are shifted

\[ \frac{(S-n)c}{2L} \quad \frac{Sc}{2L} \quad \frac{(S+1)c}{2L} \]

"higher order transverse mode frequencies"