University of California
Department of Electrical Engineering
and Computer Sciences

Prof. J. S. Smith
Fall 2004
EECS 236A

Final Exam

December 17, 2004

NAME: ____________________________  Student ID: ____________________________

Last, First

- Open book (Yariv), open notes.
- Calculators are allowed.
- Show all of your work and reasoning to receive full or partial credit.

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1) Find the spontaneous lifetime of an electron in an infinite square well potential in the shape of a cube, with a side length \( a \). The starting state is the first excited state with an extra null in the \( z \) direction. You do not have to evaluate any integrals.

\[ t_{\text{spont}} = \frac{1}{3\pi e^2 \hbar c} |r_{12}|^2 \]

(Yariv 8.3-9) where, in this case

\[ |r_{12}|^2 = |Z_{12}|^2 = |\langle 01 \parallel 11 \rangle|^2 \]

\[ = K^{-1} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \cos \left( \frac{\pi}{2a} x \right) \cos \left( \frac{\pi}{2a} y \right) \cos \left( \frac{\pi}{2a} z \right) \times \cos \left( \frac{\pi}{2a} x \right) \cos \left( \frac{\pi}{2a} y \right) \sin \left( \frac{\pi}{a} z \right) d^3r \]

where \( K = \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \cos \left( \frac{\pi}{2a} x \right) \cos \left( \frac{\pi}{2a} y \right) \cos \left( \frac{\pi}{2a} z \right) d^2r \)

\[ K = \left( \frac{a}{2} \right)^3 \int_{-a/2}^{a/2} \cos \left( \frac{\pi}{2a} z \right) \sin \left( \frac{\pi}{a} z \right) d^2r \]

\[ |r_{12}|^2 = \left( \frac{a}{2} \right)^3 \int_{-a/2}^{a/2} \cos \left( \frac{\pi}{2a} z \right) \sin \left( \frac{\pi}{a} z \right) d^2r \]

(b) For each mode, \( W_{\text{spont}} |{\hat{e}}_{kx} \rangle = \frac{a^2}{2} \]

\( \rightarrow \) spontaneous polarization in \( \hat{e}_x \) direction

(c) \( W_{\text{spont}} |{\hat{e}}_{kx} \rangle \cdot 2^1 d(\sin \theta)^2 \)
2) When we solved for the Gaussian beam shape reproduced in a resonator from the ABCD matrices, we assumed that the beam would have to reproduce itself in one round trip. Could the beam reproduce itself in two, three or more round trips? Explain thoroughly, and give examples or show why it is not possible.

For 1 round trip, we have
\[ b = \frac{Ag + D}{Cg + D} \]
\[ Cg^2 + (D-A)g - D = 0 \]

For two round trips:
\[ (A \begin{bmatrix} A & AB + DB \\ C & D \end{bmatrix}^2 = (A^2 + BC, AB + DB) \]
\[ C(A + D)g^2 + (A - D)(A + D)g - DB = 0 \]

For the general case, we use
\[ A_T = \frac{A \sin(\Theta) - \sin((2n-1)\Theta)}{\sin\Theta} \]
\[ B_T = \frac{B \sin(\Theta)}{\sin\Theta} \]
\[ C_T = \frac{C \sin(\Theta)}{\sin\Theta} \]
\[ D_T = \frac{D \sin(\Theta) - \sin((2n-1)\Theta)}{\sin\Theta} \]
\[ Cg^2 + (D_1 - A_T)g - D_T = 0 \]
\[ C \frac{g^2}{\sin\Theta} + (A - D) \frac{g^2}{\sin\Theta} - D = 0 \]
\[ Cg^2 + (A - D)g + D = 0 \Rightarrow g + \frac{D \sin\Theta}{\sin\Theta} \]

If it works for 1, it works for all.
3) The index of refraction of a semiconductor changes as the number of carriers is changed. Find an expression for the small change in the index of refraction with a small change in the number of carriers, for light which has a frequency just below that corresponding to the bandgap energy, for a semiconductor at zero temperature. Use the notation:

\[ \Delta N = \Delta N_{\text{elec}} = \Delta N_{\text{holes}} \]

\[ n \Rightarrow n + \Delta n \]

\[ m_c, m_h, x_{nc}, E_g, T \]

\[ T = \infty \]

\[ \Delta \gamma = 2 \frac{C}{k} \left( \frac{\hbar \omega - E_g}{E_g} \right)^{1/2} \Delta \omega \delta (\omega - \omega(N)) \]

\[ W(N) = \frac{E_g}{k} = \frac{k}{2m_e} \left[ \frac{\pi^2}{6} N \right]^{2/3} + E_g \]

\[ \Delta \omega = \frac{\partial \omega}{\partial N} \Delta N = \frac{k}{2m_e} \frac{2}{3} \left( \frac{\pi^2}{6} \right)^{2/3} N^{-1/3} \]

\[ \Delta X'' = -\frac{n^2}{k} 2C \left( \frac{\hbar \omega - E_g}{E_g} \right)^{1/2} \Delta \omega \delta (\omega - \omega(N)) \]

\[ \Delta X' = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X''(\omega)}{\omega - \omega'} d\omega' \]

\[ \Delta X' = \frac{1}{\pi} \frac{1}{W(N) - E_0/k} - \frac{n^2}{k} C \left( \frac{\hbar \omega - E_g}{E_g} \right)^{1/2} \Delta \omega \]

\[ n = \frac{W}{E_0} \approx \left( 1 - \frac{C_0}{2E_0} \Delta X' \right) \]

\[ \Delta N \approx \frac{C_0}{2E_0} \Delta X' \]
4) When spontaneous emission is included, the gain is just smaller than the loss in a lasing mode. For a cavity of length $L$ and mirror reflectivities both $R$, find the difference between the actual gain needed and that given by the total loss. Assume a lasing intensity $I$, and a nearly uniform mode $1$ mm in radius, and a lasing frequency $v$.

Start with the rate equation for photons:

$$\frac{dp}{dt} = P\left(\frac{\gamma}{n} - \frac{1}{c} \right) + \Sigma_{\text{spont}}$$

where

$$\gamma = \frac{(N_2 - N_1) \lambda^2 n}{8 \pi n^2 \Sigma_{\text{spont}}}$$

the spontaneous emission rate is $\frac{1}{P}$ of the stimulated emission rate, which corresponds to the $N_2$ above.

So we have

$$\frac{dp}{dt} = P\left(\frac{\gamma}{n} - \frac{1}{c} \right) + \frac{N_2 - \gamma}{(N_2 - N_1) \frac{2c}{\beta}}$$

in steady state, $\frac{dp}{dt} = 0$

$$\gamma\left(P\frac{c}{n} + \frac{N_2}{(N_2 - N_1) \frac{2c}{\beta}}\right) = \frac{P}{c}$$

$$\gamma = \frac{P c}{n} + \frac{N_2}{(N_2 - N_1) \frac{2c}{\beta}}$$

$$\gamma = \frac{1}{\frac{\beta c}{n} \frac{2}{\beta}}$$

$$P = 2\pi r^2 L \frac{n}{c}$$

$$P = \frac{1}{c} \frac{n}{p} \frac{V}{V_0}$$

$$\frac{1}{n} \left(1 - \frac{p^2}{V_0} \frac{V}{V_0} \frac{2c}{\beta} \right)$$

$$\frac{n}{c} = \frac{1}{\beta c}$$