EE236
Problem Set 2
Fall 2004
Solutions

1) Need to show
$$\langle \phi | (\hat{A}\hat{B})^{\dagger} | \psi \rangle = \langle \phi | \hat{B}^{\dagger}\hat{A}^{\dagger} | \psi \rangle$$
for arbitrary $|\phi\rangle$, $|\psi\rangle$

$$\langle \phi | \hat{B}^{\dagger}\hat{A}^{\dagger} | \psi \rangle = \langle B\phi | A^{\dagger} | \psi \rangle$$
$$= \langle A B \phi | \psi \rangle$$
$$= \langle \phi | (AB)^{\dagger} | \psi \rangle$$

2) The eigenvalues of the eigenvectors must also be real

Since the eigenvectors of $\hat{A}$ form $|\phi_n\rangle$ a complete set, we can write
$$I = \sum_n |\phi_n\rangle\langle\phi_n|$$
$$\hat{A} = \sum_m \sum_n |\phi_n\rangle\langle\phi_n|\hat{A}|\phi_m\rangle\langle\phi_m|$$
$$\hat{A} = \sum_m \sum_n |\phi_n\rangle\langle\phi_n|\lambda_m|\phi_m\rangle\langle\phi_m|$$

$$\hat{A} = \sum_m \sum_n \lambda_m \delta_{nm} |\phi_n\rangle\langle\phi_m|$$

$$\hat{A} = \sum_m \lambda_m |\phi_m\rangle\langle\phi_m|$$

$$\hat{A}^+ = \sum_m \lambda_m^* |\phi_m\rangle\langle\phi_m|$$

if λ_m is real

$$\hat{A}^+ = \hat{A}$$

3) $A|\phi_1\rangle = \lambda_1 |\phi_1\rangle$

$$A|\phi_2\rangle = \lambda_2 |\phi_2\rangle$$

$$\lambda_1 = \lambda_2$$

$$\langle\phi_1|\phi_2\rangle \neq 0$$

$|\phi_1\rangle \neq C|\phi_2\rangle$ for any C

Let $|\phi_3\rangle = |\phi_2\rangle - |\phi_1\rangle\langle\phi_1|\phi_2\rangle$

$$\hat{A}|\phi_3\rangle = \lambda_2|\phi_2\rangle - \lambda_1|\phi_1\rangle\langle\phi_1|\phi_2\rangle$$

$$= \lambda_1 |\phi_3\rangle$$

$$\langle\phi_1|\phi_3\rangle = \langle\phi_1|\phi_2\rangle - \langle\phi_1|\phi_2\rangle$$

$$= 0$$

$\cancel{+}$ $|\phi_3\rangle$ is non singular dividing by $\psi_{xn}(x)\psi_{ym}(y)\psi_{zp}(z)$

and defining the separation constant E_x E_y E_z

$$E = \underbrace{\frac{\hat{H}_x \psi_{xn}(x)}{\psi_{xn}(x)}}_{E_x} + \underbrace{\frac{\hat{H}_y \psi_{ym}(y)}{\psi_{ym}(y)}}_{E_y} + \underbrace{\frac{\hat{H}_z \psi_{zp}(y)}{\psi_{zp}(y)}}_{E_z}$$

So we have each of $\psi_{xn}(x)$ $\psi_{ym}(y)$ + $\psi_{zp}(z)$

are the eigenfunction of the 1-D H.O.

+ $E = E_x + E_y + E_z$

so $E = (n+m+p+3/2)\hbar\omega$

And the general solution is

$$\psi(x,y,z) = \sum_n \sum_m \sum_p \sqrt{\frac{\alpha^3}{\pi^{3/2}\, 2^{(n+m+p)}\, n!\, m!\, p!}} \cdot e^{-\alpha^2/2 (x^2+y^2+z^2)} \cdot H_n(\alpha x)\, H_m(\alpha y)\, H_p(\alpha z)$$

4) Ya-iv 2,1

The Hamiltonian for a 3-D oscillator is:

$$\hat{H} = -\frac{\hbar^2 \nabla^2}{2m} + \frac{1}{2} K(x^2 + y^2 + z^2)$$

$$= \left(-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}Kx^2\right) + \left(-\frac{\hbar}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}Ky^2\right) + \left(-\frac{\hbar}{2m}\frac{\partial^2}{\partial z^2} + \frac{1}{2}Kz^2\right)$$

$$\hat{H} = \hat{H}_x + \hat{H}_y + \hat{H}_z$$

We can use separation of variables to try to find eigenfunctions of the form

$$\psi(x,y,z) = \psi_{xn}(x)\,\psi_{ym}(y)\,\psi_{zp}(z)$$

but that does not mean that the general solution must be the product of separate functions of 1 variable

$$\hat{H}\,\psi_{nmp} = \hat{H}_x \psi_{xn}(x)\,\psi_{ym}(y)\,\psi_{zp}(z) + \hat{H}_y \psi_{xn}(x)\,\psi_{ym}(y)\,\psi_{zp}(z) + \hat{H}_z \psi_{xn}(x)\,\psi_{ym}(y)\,\psi_{zp}(z)$$

$$= E\,\psi_{xn}(x)\,\psi_{ym}(x)\,\psi_{zp}(z)$$

5) Yariv 2.3

If you did not use raising and lowering operators for this one, you earned your pain!

$$\int_{-\infty}^{\infty} u_n^*(x)\, x^2\, dx$$

$$= \int_{-\infty}^{\infty} u_n(x)\;\; x^2\; u_n(x)\, dx$$

$$= \langle u_n |\, x^2 \,| u_n \rangle$$

$$\hat{x} = \frac{1}{\sqrt{2}\,\alpha}\,(a^+ + a)$$

$$\hat{x}\,\hat{x} = \frac{1}{2\alpha^2}\left(a^+a^+ + a^+a + aa^+ + aa\right)$$

only the terms raising & lowering the same number of times will not be orthogonal

$$\langle u_n | x^2 | u_n \rangle = \frac{1}{2\alpha^2} \langle u_n | (a^+a + aa^+) | u_n \rangle$$

$$a^+a\,|u_n\rangle = n\,|u_n\rangle$$

$$[a, a^+] = a\,a^+ - a^+a = 1$$

$$a\,a^+ = a^+a + 1$$

$$a\,a^+ | u_n\rangle = \frac{1}{2\alpha^2} \langle u_n | 2a^+a - 1 | u_n\rangle$$

$$= \frac{1}{2\alpha^2}\,(2n+1)$$

$$\int_{-\infty}^{\infty} x^2 \, u_{n+2} u_n \, dx$$

$$= \langle u_{n+2} | \hat{x}^2 | u_n \rangle$$

$$= \langle u_{n+2} | \frac{1}{2\alpha^2} (a^\dagger a^\dagger + a^\dagger a + a a^\dagger + a a) | u_n \rangle$$

only the $a^\dagger a^\dagger$ term is non-orthogonal

$$= \langle u_{n+2} | \frac{1}{2\alpha^2} a^\dagger a^\dagger | u_n \rangle$$

$$= \langle u_{n+2} | \frac{1}{2\alpha^2} a^\dagger \sqrt{n+1} \, | u_{n+1} \rangle$$

$$= \langle u_{n+2} | \frac{1}{2\alpha^2} \sqrt{n+2} \sqrt{n+1} \, | u_{n+2} \rangle$$

$$= \frac{1}{2\alpha^2} \sqrt{n+2} \sqrt{n+1}$$

6 Yariv problem 2.6

$$a \equiv \frac{\alpha}{\sqrt{2}} x + \frac{1}{\sqrt{2}\alpha}\frac{\partial}{\partial x}$$

$$a^\dagger \equiv \frac{\alpha}{\sqrt{2}} x - \frac{1}{\sqrt{2}\alpha}\frac{\partial}{\partial x}$$

The Hamiltonian is

$$\hat{H} = \frac{p_x^2}{2m} + \frac{1}{2}Kx^2$$

$$= \hbar\omega\left(a^+ a + \frac{1}{2}\right) \qquad (2.2\text{-}30)$$

and we have $[a, a^+] = 1$ (2.2-28)

$$a a^+ = a^+ a + 1$$

$$\hat{H} a^+ |u_n\rangle = \hbar\omega\left(a^+ a a^+ + \frac{1}{2}a^+\right)|u_n\rangle$$

$$= \hbar\omega\left(a^+(a^+ a + 1) + \frac{1}{2}a^+\right)|u_n\rangle$$

$$= a^+\hbar\omega\left(a^+ a + \frac{3}{2}\right)|u_n\rangle$$

$$= a^+(\hat{H} + \hbar\omega)|u_n\rangle$$

$$= a^+(E_n + \hbar\omega)|u_n\rangle$$

$$= (E_n + \hbar\omega)\, a^+|u_n\rangle$$

$\rightarrow$ so $a^+|u_n\rangle$ is an eigenstate of $\hat{H}$

with the eigenvalue $E_n + \hbar\omega = E_{n+1}$

So we know $a^\dagger |u_n\rangle = C |u_{n+1}\rangle$ (*)

to find the value of C, find
$\| a^\dagger |u_n\rangle \| = \| C |u_{n+1}\rangle \|$.

$$\langle u_n | a a^\dagger | u_n \rangle = |C_n|^2 \langle u_{n+1} | u_{n+1} \rangle$$

$$= \langle u_n | a^\dagger a + 1 | u_n \rangle = |C_n|^2$$

$$= \langle u_n | n + 1 | u_n \rangle = |C_n|^2$$

$$|C_n|^2 = |n+1|$$

$$C_n = (n+1)^{1/2} e^{i\theta}$$

where θ is an arbitrary choice
examining 2.2-14, we can
see that the choice Yariv made
was $\theta = 0$.

$$\Rightarrow a^\dagger |u_n\rangle = \sqrt{n+1}\, |u_{n+1}\rangle$$

from (*) above, $a a^\dagger |u_n\rangle = C a |u_{n+1}\rangle$

$$(a^\dagger a + 1) |u_n\rangle = (n+1)^{1/2} a |u_{n+1}\rangle$$

$$(n+1) |u_n\rangle = (n+1)^{1/2} a |u_{n+1}\rangle$$

$m \rightarrow$ $$m |u_{m-1}\rangle = m^{1/2} a |u_m\rangle$$

$a|u_n\rangle = n^{1/2}|u_{n-1}\rangle$

(if u_{n-1} exists)

7) Yariv 2.7

$$G(s,\xi) = e^{-s^2+2s\xi}$$

(1) $$= \sum_{n=0}^{\infty} [H_n(\xi)/n!]\, s^n$$

take the derivitives of both sides with respect to ξ

$$2s e^{-s^2+2s\xi} = \sum_{n=0}^{\infty} \left[\frac{\partial H_n(\xi)}{\partial \xi}\right] \frac{1}{n!} s^n$$

(2) $$e^{-s^2+2s\xi} = \sum_{n=0}^{\infty} \left[\frac{\partial H_n(\xi)}{\partial \xi}\right] \frac{1}{2n!} s^{n-1}$$

$$= \sum_{m=0}^{\infty} \frac{\partial H_{m+1}(\xi)}{\partial \xi} \frac{1}{2(m+1)!} s^m$$

Since (1) & (2) are equal, the coefficients of s^n must each be equal

$$\frac{H_n(\xi)}{n!} = \frac{\partial H_{n+1}(\xi)}{\partial \xi} \frac{1}{2(n+1)n!}$$

$$\frac{\partial H_{n+1}(\xi)}{\partial \xi} = 2(n+1)\, H_n(\xi)$$

changing variables

$$\frac{\partial H_n(\xi)}{\partial \xi} = 2n\, H_{n-1}(\xi)$$

$$G(s,\xi) = e^{-s^2+2s\xi} = \sum_{n=0}^{\infty} \left[H_n(\xi)/n!\right] s^n$$

take $\frac{\partial}{\partial s}$ of both sides

$$e^{-s^2+2s\xi}(-2s+2\xi) = \sum_{n=0}^{\infty} H_n(\xi)\frac{n}{n!} s^{n-1}$$

substituting for $e^{-s^2+2s\xi}$

$$\sum_{n=0}^{\infty} H_n(\xi)\frac{1}{n!}\left(-2s^{n+1}+2\xi s^n\right)$$

$$= \sum_{n=0}^{\infty} H_n(\xi)\frac{1}{(n-1)!} s^{n-1}$$

$$\sum_{m=0}^{\infty} H_{m-1}(\xi)\frac{-2}{(m-1)!} s^m + \sum_{n=0}^{\infty} H_n(\xi)\frac{1}{n!} 2\xi s^n$$

$$= \sum_{p=0}^{\infty} H_{p+1}(\xi)\frac{1}{p!} s^p$$

taking the terms for equal coefficients of ξ^n equal again

$$H_{n-1}(\xi)\frac{-2}{(n-1)!} + H_n(\xi)\frac{1}{n!}2\xi = H_{n+1}(\xi)\frac{1}{n!}$$

$$H_{n-1}(\xi)(-2n) + H_n(\xi)\,2\xi = H_{n+1}(\xi)$$

$$\boxed{\xi H_n(\xi) = \frac{1}{2}H_{n+1}(\xi) + nH_{n-1}(\xi)}$$