Fall 2004
EE236
Problem set 3
Solutions

2.4

$$\langle p^2/2m \rangle = \langle u_n | \hat{p}^2/2m | u_n \rangle$$

$$\hat{p}^2 = \left(\frac{i\hbar\alpha}{\sqrt{2}} (a^\dagger - a) \right)^2$$

$$= -\frac{\hbar^2\alpha^2}{2} \left(a^\dagger a^\dagger - a^\dagger a - a a^\dagger + a a \right)$$

$$\langle p^2/2m \rangle = + \langle u_n | \frac{\hbar^2\alpha^2}{2m} \left(a^\dagger a + a a^\dagger \right) | u_n \rangle$$

$$= + \langle u_n | \frac{\hbar^2\alpha^2}{2m} \left(a^\dagger \sqrt{n} \right) | u_{n-1} \rangle$$

$$+ \langle u_n | \frac{\hbar^2\alpha^2}{2m} \left(a \sqrt{n+1} \right) | u_{n+1} \rangle$$

$$= \frac{\hbar^2\alpha^2}{2m} \left(n + \left(\sqrt{n+1} \right)^2 \right)$$

$$= \frac{\hbar^2\alpha^2}{2m} (2n+1)$$

$$= \frac{1}{2} \left(\frac{\hbar^2\alpha^2}{2m} \right) (n + 1/2)$$

$$= 1/2 \, \hbar\omega_0 (n + 1/2)$$

half of the total Energy

2.5)

$$\langle \Delta x^2 \rangle = \langle (X - \langle x \rangle)^2 \rangle$$

$$= \langle X^2 - 2x\langle x \rangle + \langle x \rangle^2 \rangle$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle \Delta p^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2$$

$$\langle x \rangle = 0 \quad \text{and} \quad \langle p \rangle = 0$$

for the energy eigenstates

$$\langle x^2 \rangle = \frac{2n+1}{2\alpha^2}$$ (from prob 2.3)

$$\langle p^2 \rangle = \hbar^2 \alpha^2 (2n+1)$$ (from prob 2.4)

$$\sqrt{\langle \Delta x^2 \rangle}\sqrt{\langle \Delta p^2 \rangle} = \tfrac{1}{2}\hbar(2n+1)$$

3) Yariv 2.9

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

$$= -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}K_x x^2 + \frac{1}{2}K_y y^2 + \frac{1}{2}K_z z^2$$

$$= \left(\frac{p_x^2}{2m} + \frac{1}{2}K_x x^2\right) + \left(\frac{p_y^2}{2m} + \frac{1}{2}K_y y^2\right) + \left(\frac{p_z^2}{2m} + \frac{1}{2}K_z z^2\right)$$

$$= \hbar w_x(a_x^+ a_x + 1/2) + \hbar w_y(a_y^+ a + 1/2) + \hbar w_z(a_z^+ a + 1/2)$$

4) $\hat{H} = \begin{pmatrix} E_1 & \mu \\ \mu & E_2 \end{pmatrix}$ over $|1\rangle$ $|2\rangle$

to solve this exactly, find the eigen states of $\hat{H}$

the characteristic equation is

$$\begin{vmatrix} E_1-\lambda & \mu \\ \mu & E_2-\lambda \end{vmatrix}$$

$$(E_1-\lambda)(E_2-\lambda)-\mu^2=0$$

$$E_1E_2-E_1\lambda-E_2\lambda+\lambda^2-\mu^2=0$$

$$+\lambda^2-(E_1+E_2)\lambda+(E_1E_2-\mu^2)=0$$

$$\lambda_{1,2}=+\frac{E_1+E_2}{2}\pm\sqrt{\frac{(E_1+E_2)^2}{4}-(\mu^2-E_1E_2)}$$

$$\lambda_{1,2}=+\frac{E_1+E_2}{2}\pm\sqrt{\frac{(E_1-E_2)^2}{4}-\mu^2}$$

so the period is $\frac{2\pi}{\Delta\omega} = \frac{2\pi\hbar}{\Delta E}$

$$\tau = 2\pi\hbar\left((E_1-E_2)^2-4\mu\right)^{-1/2}$$

for μ small

$$\tau \approx 2\pi\hbar\frac{1}{|E_1-E_2|}$$

we now find an expression for the eigenstates of H in terms of $|1\rangle, |2\rangle$

$$\begin{pmatrix} E_1 & \mu \\ \mu & E_2 \end{pmatrix}|\phi_1\rangle = \left(\frac{E_1+E_2}{2} + \sqrt{\frac{(E_1-E_2)^2}{4} - \mu^2}\right)|\phi_1\rangle$$

$$\begin{pmatrix} E_1 & \mu \\ \mu & E_2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \left(\frac{E_1+E_2}{2} + \sqrt{\frac{(E_1-E_2)^2}{4} - \mu^2}\right)\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

$$E_1 a_1 + \mu b_1 = \left(\frac{E_1+E_2}{2} + \sqrt{\frac{(E_1-E_2)^2}{4} - \mu^2}\right) a_1$$

$$C_\pm = \frac{b_1}{a_1} = \left[\frac{\left(\frac{E_2-E_1}{2}\right) \pm \sqrt{\frac{(E_1-E_2)^2}{4} - \mu^2}}{\mu}\right]$$

So we get the two eigenstates

$$|\phi_\pm\rangle = \frac{1}{\sqrt{1+c_\pm^2}}\begin{pmatrix}1\\ c_\pm\end{pmatrix}$$

at $t=0$, we have the initial condition $\psi(t=0) = |1\rangle$

so

$$A|\phi_+\rangle + B|\phi_-\rangle = |1\rangle$$

$$A\frac{1}{\sqrt{1+C_+^2}}C_+^2 + B\frac{1}{\sqrt{1+C_-^2}}C_-^2 = 0$$

$$A = -\frac{\sqrt{1+C_+^2}}{C_+^2}\frac{C_-^2}{\sqrt{1+C_-^2}}B$$

and $|\psi(t)\rangle = A e^{-i\lambda_+ t}|\phi_+\rangle + B e^{-i\lambda_- t}|\phi_-\rangle$

and the maximum occupancy of state 2 occurs when the Δ phase is 180

we also have $A\frac{1}{\sqrt{1+C_+^2}} + B\frac{1}{\sqrt{1+C_-^2}} = 1$

$$B = \sqrt{1+C_+^2}\left(1 - \frac{A}{\sqrt{1+C_+^2}}\right)$$

$$B = \left(\sqrt{1-C_-^2} - \sqrt{1+C_+^2}\,\frac{C_-^2}{C_+^2}\,B\right)$$

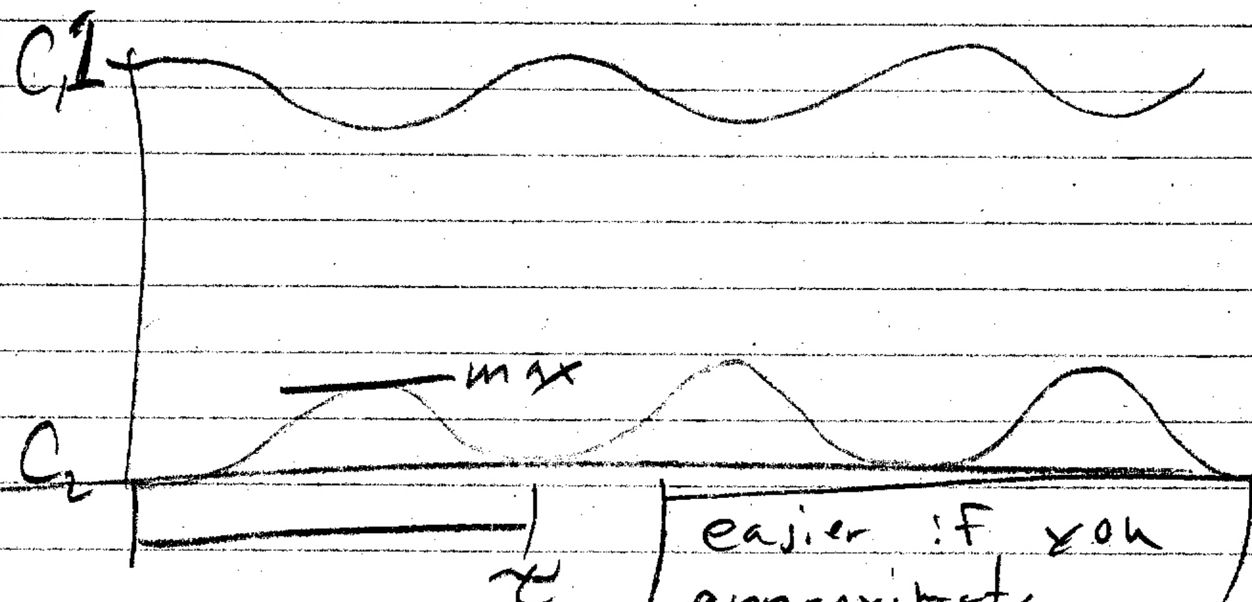
$$B\left(1 + \sqrt{1+C_+^2}\,\frac{C_-^2}{C_+^2}\right) = \sqrt{1-C_-^2}$$

$$B = \frac{\sqrt{1-C_-^2}}{\left(1+\sqrt{1+C_+^2}\left(\frac{C_-^2}{C_+^2}\right)\right)}$$

so the maximum amplitude to be in $|2\rangle$ is $2B$

$$\mathrm{Max}(|2\rangle) = \frac{2\sqrt{1-C_-^2}}{\left(1+\sqrt{1+C_+^2}\left(\frac{C_-^2}{C_+^2}\right)\right)}$$

where C_- & C_+ are defined above



easier if you approximate