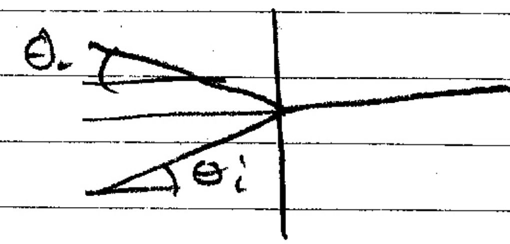
EE236
Problem set 5
solution



1) We have an incident wave

$$\vec{E}_i(\vec{r},t) = \vec{E}_i e^{i(k_i \cdot r - \omega t)}$$

reflected wave

$$E_r(\vec{r},t) = \vec{E}_r e^{i(k_r \cdot r - \omega t)}$$

and transmitted

$$E_t(\vec{r},t) = \vec{E}_t e^{i(k_t r - \omega t)}$$

at the boundary take $z=0$

$$E_{t_1} = E_{t_2}$$

$$E_{ix} e^{i(k_i \cdot r - \omega t)} + E_{rx} e^{i(k_r \cdot r - \omega t)} = E_{tx} e^{i(k_t \cdot r - \omega t)} \quad (z=0)$$

since the functions $e^{ik\cdot g}$ are orthogonal and these must hold for all $x, y,$ and t

$$k_{ix}x + k_{iy}y - \omega_i t = k_{rx}x + k_{ry}y - \omega_r t$$

$$= k_{tx}x + k_{ty}y - \omega_t t$$

$\Rightarrow \omega_i = \omega_r = \omega_t$

$k_{ix\,i} = k_{rx\,r} = k_t \qquad k \quad = k_{ry} = k_{tz}$

②

and we have $\frac{\omega}{|K|} = \frac{c}{n}$

$$|K|^2 = K_x^2 + K_y^2 + K_z^2 = \omega \frac{n}{c}$$

so in media 1 $K_{rz} = \pm K_z$

since the reflected beam is not the one incident on the surface,

$$K_{rz} = -K_{iz}$$

$$\theta_r = \sin \frac{K_{rz}}{|K_r|} = \sin \frac{K_{iz}}{|K_i|}$$

$$\theta_r = \theta_i$$

for the transmitted ray

$$K_{tz} = K_t^2 - K_{tx}^2 - K_{ty}^2$$

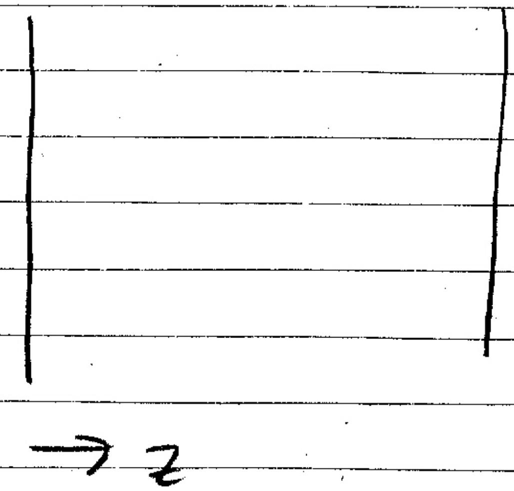
$$K_{iz} = K_i^2 - K_{ix}^2 - K_{iy}^2$$

$$\sin^2\theta_t = \frac{K_{tx}^2 + K_{ty}^2}{K_t^2} = \frac{K_{ix}^2 + K_{iy}^2}{\frac{n_2^2}{n_1^2} K_i^2}$$

$$\sin^2\theta_t = \left(\sin^2\theta_i\right)\left(\frac{n_1}{n_2}\right)^2$$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2}$$

2)



Starting with the wave equation

$$\nabla^2 E - \mu \varepsilon \frac{\partial^2}{\partial t^2} E = 0$$

$$\nabla^2 H - \mu \varepsilon \frac{\partial^2}{\partial t^2} H = 0$$

$$D = \varepsilon E$$

$$\nabla \times E = -\frac{\partial}{\partial t} \mu_0 H \quad \text{(nonmagnetic)}$$

setting $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = 0$

$$\frac{\partial^2}{\partial x^2} E(z,t) - \mu \varepsilon \frac{\partial^2}{\partial t^2} E(z,t) = 0$$

separating variables $E = E(0)\, e^{-i\omega t}$

$$\frac{\partial^2}{\partial x^2} E(z) + \omega^2 \mu \varepsilon E(z) = 0$$

(4)

$$E(z) = \vec{E}_1 \sin(kz) + \vec{E}_2 \cos(kz)$$

since $\vec{E} \times \vec{K} = 0$, and the tangential fields at a conductor are zero, (B.C. at $z = 0$)

$$\vec{E}(z) = (\vec{E}_x + \vec{E}_y) \sin(kz)$$

taking the separation to be a

$$\vec{E}(a) = 0 = \sin ka$$

$$\Rightarrow K = \frac{m\pi}{a}$$

$$\boxed{\vec{E}(z) = R_e\{(\vec{E}_y + E_x) \sin\left(\frac{m\pi}{a} z\right) e^{-i\omega t}\}}$$

Since E only depends on z

$$\nabla \times \vec{E} = \left(-\frac{E_y}{\partial z}\right) \hat{x} + \left(\frac{\partial E_x}{\partial z}\right) \hat{y}$$

$$-\frac{\partial}{\partial t} \mu_0 H = \nabla \times E = -E_y \hat{x} \frac{m\pi}{a} \sin\left(\frac{m\pi}{a} z\right) + E_y \frac{m\pi}{a} \sin\left(\frac{m\pi}{a} z\right)$$

$$H = \frac{+1}{\mu_0} \int \left(E_y \frac{m\pi}{a} \hat{x} - E_x \frac{m\pi}{a} \hat{y}\right) \cos\left(\frac{m\pi}{a} z\right) e^{-i\omega t}$$

$$H = \frac{1}{\mu_0} \left(E_y \frac{m\pi}{a} \hat{x} - E_x \frac{m\pi}{a} \hat{y}\right) \cos\left(\frac{m\pi}{a}\right) \left(\frac{1}{-i\omega}\right) e^{-i\omega t}$$

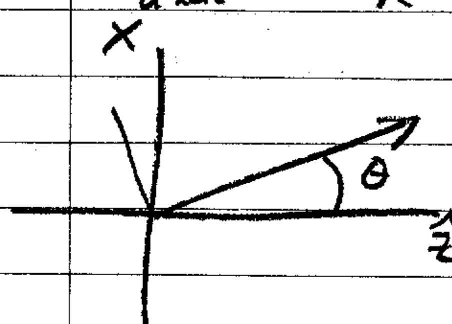
$$H = R_e \left\{ \frac{-1}{i\omega\mu_0} \frac{m\pi}{a} \left[E_y \hat{x} - E_x \hat{y}\right] \cos\left(\frac{m\pi}{a}\right) e^{-i\omega t} \right\}$$

3) Yariv 5.10

Find the direction of power flow
$\vec{E} \times \vec{H}$ in a uniaxial crystal

a) for an ordinary ray,

Since propagation in a uniaxial crystal is symmetric about the optical axis, we can rotate the problem about the optical axis to put $\hat{z}$ on the optical axis and K in the $x-z$ plane



Since $\vec{D} \times \vec{K} = 0$
we have two polarizations
$D \parallel \hat{y}$ (ordinary)

D in the $x-z$ plane
(extraordinary)

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{xx} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

so for the ordinary ray E is also $\parallel \hat{y}$

and $\vec{E} \times \vec{H}$ is $\parallel \vec{K}$

(b) for the extraordinary ray

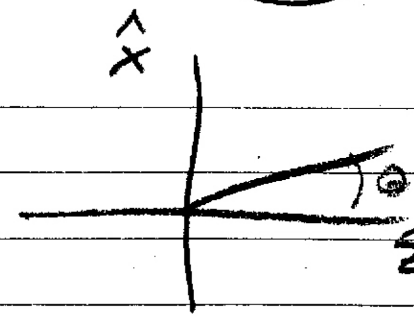
$$D = \left(-\sin\theta\,\hat{z} + \cos\theta\,\hat{x}\right)|D|$$

$$\vec{E} = \left(-\frac{1}{\varepsilon_{zz}}\sin\theta\,\hat{z} + \frac{1}{\varepsilon_{xx}}\cos\theta\,\hat{x}\right)|D|$$

since $\vec{E} \times \vec{H}$ is $\perp E$,
the direction of $\vec{E} \times \vec{H}$ is

$$\tan\theta_{E \times H} = \frac{\frac{1}{\varepsilon_{zz}}\sin\theta}{\frac{1}{\varepsilon_{xx}}\cos\theta}$$

$$\theta_{E \times H} = \tan^{-1}\left[\frac{\varepsilon_o}{\varepsilon_e}\tan\theta\right]$$



4) $V_g(\theta) = \nabla_k \omega(K)$

$$K^2 = \omega^2\left(\frac{n}{c}\right)^2$$

$$\omega^2 = K^2\left(\frac{c}{n}\right)^2$$

but n is a function of $\vec{K}$

$$\Theta = \tan^{-1}\frac{K_x}{K_z}$$

$$\omega^2 = K^2c^2\left(\frac{\cos^2\Theta}{n_o^2} + \frac{\sin^2\Theta}{n_e^2}\right)$$

$$\omega^2 = K^2C^2\left(\frac{K_z^2}{n_o K^2} + \frac{K_x^2}{n_e K^2}\right)$$

$$\omega^2 = C^2\left(K_z^2\frac{1}{n_o^2} + K_x^2\frac{1}{n_e^2}\right)$$

$$\omega = c\left(\frac{K_z^2}{n_o^2} + \frac{K_x^2}{n_e^2}\right)^{1/2}$$

$$\nabla_{\vec{k}}\omega = \hat{x}\frac{\partial}{\partial K_x}\omega + \hat{y}\frac{\partial}{\partial K_y}\omega$$

$$\nabla_{\vec{k}}\omega = \hat{x}\,2\frac{K_x}{n_e^2}\,\frac{1}{2}\left(\frac{K_z^2}{n_o^2} + \frac{K_x^2}{n_e^2}\right)^{-1/2}$$
$$+\hat{z}\,2\frac{K_z}{n_o^2}\,\frac{1}{2}\left(\frac{K_z^2}{n_o^2} + \frac{K_x^2}{n_e^2}\right)^{-1/2}$$

direction of $\nabla_{\vec{k}}\omega = \tan^{-1}\left[\frac{n_o^2}{n_e^2}\tan\frac{K_z}{K_x}\right]$

$$= \tan^{-1}\left[\frac{n_o^2}{n_e^2}\tan\Theta\right]$$

$\Rightarrow$ same as power flow