EE 276
Problem set 6
Solution

1) The approximate mode
is a standing wave, so
we can use Yariv 5.6-14

\[ E_x(r,t) = -\delta \frac{\omega x}{V_s} \left[ a_t^+ - a_r \right] \sin(k x) \]

and the state of the mode
is \( | n \rangle \)

so the RMS \( E \) field is

\[ \sqrt{\langle |E|^2 \rangle} \]

\[ = \left( \frac{\hbar \omega_c}{V_s} \right)^{1/2} \left( \langle n \langle a_t^+ - a_r \rangle^2 \right)^{1/2} \frac{1}{\sin(k x z)} \]

\[ = \left( \frac{\hbar \omega_c}{V_s} \right)^{1/2} \left( \langle n \langle a_t^+ a_r + a_r^+ a_t \rangle \right)^{1/2} \frac{1}{\sin(k x z)} \]

\[ = \left( \frac{\hbar \omega_c}{V_s} \right)^{1/2} \left( 2 n + 1 \right)^{1/2} \frac{1}{\sin(k x z)} \]

\( \Rightarrow \) the peak occurs when \( \sin(k x z) = 1 \)
2) For a single lens followed by a distance \(d\), we have

\[
\left. \begin{array}{c}
\frac{1}{r_{\text{out}}} = \left| \frac{1}{r_{\text{out}}} \right| = \left| \frac{1}{-\frac{1}{f}} - \frac{d}{f} \right| = \left| r_{\text{in}} \right|
\end{array} \right|
\]

From (6.1-10) we get

\[r_{s+1} - 2br_{s+1} + r_s = 0\]

where \(b = \frac{1}{2}(A+D) = \frac{1}{2}(2 - \frac{D}{f})\)

\[b = (1 - \frac{d}{2f})\]

Substituting \(r_s = pe^{i\theta}\)

\[e^{2i\theta} - 2be^{i\theta} + 1 = 0\]

\[e^{i\theta} = b \pm i\sqrt{1 - b^2}\]

\[\cos \theta = \frac{1}{2}(A+D)\]

and \(r_s = r_{\text{max}} \sin (5\theta + \delta)\)

For stability we need \(\cos \theta\) real, so \(-1 \leq \frac{1}{2}(A+D) \leq 1\)
\[-1 \leq \left(1 - \frac{d}{2f}\right) \leq 1\]

\[-1 \leq \left(\frac{d}{2f} - 1\right) \leq 1\]

\[0 \leq \frac{d}{2f} \leq 2\]

\[0 < d \leq 4f\]

\[\cos \theta = \frac{1}{2}(A + D) = \left(1 - \frac{d}{2f}\right) + \frac{1}{2}\]

\[r_0 = r_{\text{max}} \sin(\theta)\]

\[r_{\text{max}} = \frac{r_0}{\sin \theta}\]

\[r_1 = r_0 + dr_0\]

\[r_1 = r_{\text{max}} \sin(\theta + \delta)\]

\[r_0 + dr_0' = \frac{r_0}{\sin \delta} \sin(\theta + \delta)\]

\[s' = \sin(\delta)(1 + \frac{dr_0'}{r_0}) = \sin(\theta + \delta)\]

\[\sin \delta(1 + \frac{dr_0'}{r_0}) = \sin \theta \cos \delta + \cos \theta \sin \delta\]


\[ (1 + \frac{d r_0'}{r_0}) = \sin \theta \frac{1}{\tan \delta} + \cos \theta \]

Since \( \cos \theta = \frac{1}{2} (A+D) \)

\[ \sin \theta = \sqrt{1 - \cos^2 \theta} \]

\[ = \sqrt{1 - \left[ \frac{1}{2} (A+D) \right]^2} \]

\[ (1 + \frac{d r_0'}{r_0}) = \sqrt{1 - \frac{1}{4} (A+D)^2} \frac{1}{\tan \delta} + \frac{1}{2} (A+D) \]

\[ (1 + \frac{d r_0'}{r_0}) = \frac{1}{2} (A+D) = \sqrt{1 - \frac{1}{4} (A+D)^2} \frac{1}{\tan \delta} \]

\[ \tan \delta \left[ (1 + \frac{d r_0'}{r_0}) - \frac{1}{2} (2 - \frac{d}{r}) \right] = \sqrt{1 - \frac{1}{4} (2 - d/r)^2} \]

\[ \tan \delta \frac{\sqrt{1 - \frac{1}{4} (2 - d/r)^2}}{\left[ (1 + \frac{d r_0'}{r_0}) - \frac{1}{2} (2 - \frac{d}{r}) \right]} \]

\[ \tan \delta = \frac{\sqrt{4 - 4 + 4 \frac{d}{r} - \left( \frac{d}{r} \right)^2}}{\left[ 2 (1 + \frac{d r_0'}{r_0}) - (2 - \frac{d}{r}) \right]} \]
\[ \tan \delta = \frac{\frac{d}{f} \sqrt{4 \frac{F}{d} - 1}}{2 \frac{d}{r_0} \frac{r_1}{r_0} + \frac{d}{f}} \]

\[ \tan \delta = \frac{\sqrt{4 \frac{F}{d} - 1}}{2 \frac{r_1}{r_0} + 1} \]

\[ r_{\text{max}} = \frac{r_0}{\sin \delta} \]

\[ \sin \delta = \frac{\tan \delta}{\sqrt{1 + \tan^2 \delta}} \]

\[ r_{\text{max}} = r_0 \frac{1}{\tan \delta} \]

\[ r_{\text{max}} = r_0 \left[ \frac{1}{(\tan \delta)^2 + 1} \right] \]

\[ r_{\text{max}} = r_0 \left[ \frac{(2 \frac{r_1}{r_0} + 1)^2}{4 \frac{f}{d} - 1} + 1 \right] \]

\[ r_{\text{max}} = r_0^2 \left[ \frac{4 \frac{f^2}{d} + 4 \frac{r_1^2}{r_0^2} + 1 + 4 \frac{f}{d}}{4 \frac{f}{d} - 1} \right] \]
\[ r_{\text{max}}^2 = r_0^2 \left[ \frac{4f \left( \frac{r_0'}{r_0} \right)^2 + 4f \frac{r_0'}{r_0} + 4f}{4f - d} - 1 \right] \]

\[ = r_0^2 \left[ \frac{4f \left( \frac{r_0'}{r_0} \right)^2 + 4f \frac{r_0'}{r_0} + 1}{4f - d} \right] \]

\[ = \frac{(4f) \left( d + r_0^2 + dr_0 r_0' + r_0' \right)}{(4f - d)} \]

\[ r_{\text{max}}^2 = \frac{4f}{(4f - d)} \left( r_0^2 + dr_0 r_0' + dr_0' \right) \]
6.3

An ideal thin lens will image a plane wave (from an object at \( \infty \)) onto a point. It does this by making the total phase change equal for all paths, so that the light will interfere constructively.

\[
\Delta \phi = \frac{\sqrt{r^2 + f^2}}{\lambda/n} = \frac{k}{2\pi} \sqrt{f^2 + r^2}
\]
So to get constructive interference the lens must provide the opposite phase shift needed. 

\[ E_R(x,y) = E_L(x,y) \exp \left[ +ik \sqrt{r^2 + x^2 + y^2} \right] \]

where \( r^2 = x^2 + y^2 \)

\[ E_R(r) = E_L(r) \exp \left[ +ik \left( r^2 \left( 1 + \frac{r^2}{4f^2} \right) \right) \right] \]

the paraxial approximation is \( r \ll f \)

\[ E_R(r) \approx E_L(r) \exp \left[ +ikf \left( 1 + \frac{r^2}{2f^2} + \ldots \right) \right] \]

the overall phase shift does not make a difference so

\[ E_R(r) = E_L(r) \exp \left[ +ik \frac{r^2}{f} \right] \approx E_L(r) \exp \left[ +ik \frac{x^2 + y^2}{f} \right] \]
6.6 \[ r_{i+1} = Ar_i + Br_i' \]
\[ r_i' = Cr_i + Dr_i' \]

(1) Homogeneous medium of length d
The slope does not change, so \( C = 0 \) and \( D = 1 \)
If the ray's position is changed, the output changes by the same amount, so \( A = 1 \)

The change in the radius is slope-dictated.
so \( B = d \)

\[ \Rightarrow \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \]

(2) Thin lens: the output is the same radius as the input, so \( A = 1 \) and \( B = 0 \)
A ray going through the center is not changed in its slope, so \( D = 1 \)
A ray with \( r_i' = 0 \) goes through the focus, so \( r_{i+1} = -\frac{r_i}{4} \), so \( C = -\frac{1}{4} \)
\[ \begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{pmatrix} \]
\( r_{i+1} = r_i \quad \text{so} \quad A = 1 \quad \text{and} \quad B = 0 \)

change in slope is independent of \( r \), so \( C = 0 \)

We then have

\[
\frac{\sin \theta_{i+1}}{\sin \theta_i} = \frac{N_1}{N_2}
\]

using the per-axial approximation

\[
\sin \theta \approx \theta = r'
\]

\[
\frac{r_{i+1}}{r_i} = \frac{N_1}{N_2} = D = \frac{N_1}{n_2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}
\]

\( \text{(4)} \) For a curved dielectric interface, the position does not change \( r \) so \( A = 1 \quad B = 0 \)

From Snell's law

\[
\frac{\sin \theta_{i+1}}{\sin \theta_i} = \frac{N_1}{N_2}
\]
but since the interface is not flat, we need to calculate $\Theta$

the radius $R$ is $1$ to the interface so $\Theta$ is the difference between the slope of that line and $r'$

$\Theta_i = r'_i - \frac{r_i}{R}$

so we have $r_{i+1}' - \frac{r_0}{R} = \frac{n_1}{n_2}$

$\frac{r_i'}{r_i} - \frac{r_i}{R} = \frac{n_1}{n_2} \left( r_i' - \frac{r_i}{R} \right)$

$r_{i+1}' = \frac{n_1}{n_2} (r_i') + \left( \frac{n_1}{n_2} + 1 \right) \frac{r_i}{R}$

$\Rightarrow C = \left( -\frac{n_1}{n_2} + 1 \right) \frac{1}{R} = \frac{n_2 - n_1}{n_1} \frac{1}{R}$

$D = \frac{n_1}{n_2} \begin{pmatrix} 1 & 0 \\ \frac{n_2 - n_1}{R} & n_1 \end{pmatrix}$
5) the radius does not change, so again we have
\[ A = 1 \quad \text{and} \quad B = 0 \]

We now have \( \theta_i = \theta_{i+1} \), from the normal to the surface.

The normal has the same slope as the radius line \( R \).

So \( \theta_i = r'_i - \frac{r_0}{R} \)

\[ \theta_{i+1} = r'_{i+1} + \frac{r_0}{R} \]

\[ r'_{i+1} + \frac{r_0}{R} = r'_i - \frac{r_0}{R} \]

\[ r'_{i+1} = r'_i - 2 \frac{r_0}{R} \]

\[
\begin{pmatrix}
1 & 0 \\
-\frac{2}{R} & 1
\end{pmatrix}
\]
6) this is directly from 
6.4-5 for a media 
of length \( l \)

\[
\begin{align*}
\mathbf{r}(l) &= \cos \left( \frac{\sqrt{k_2}}{k} l \right) \mathbf{r}_0 + \sqrt{\frac{k}{k_2}} \sin \left( \frac{\sqrt{k_2}}{k} l \right) \mathbf{r}_1 \\
\mathbf{r}'(l) &= -\sqrt{\frac{k_2}{k}} \sin \left( \frac{\sqrt{k_2}}{k} l \right) \mathbf{r}_0 + \cos \left( \frac{\sqrt{k_2}}{k} l \right) \mathbf{r}_1 \\
&= \begin{pmatrix} \cos \left( \frac{\sqrt{k_2}}{k} l \right) & \frac{\sqrt{k_2}}{k} \sin \left( \frac{\sqrt{k_2}}{k} l \right) \\ -\frac{\sqrt{k_2}}{k} \sin \left( \frac{\sqrt{k_2}}{k} l \right) & \cos \left( \frac{\sqrt{k_2}}{k} l \right) \end{pmatrix}
\end{align*}
\]
6.8(a)

the ABCD matrix
to a distance \( l_3 > l_2 \)
from the waist position is

\[
\begin{pmatrix}
1 & l_3-l_2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & \frac{1}{n}
\end{pmatrix}
\begin{pmatrix}
1 & l_2-l_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & n
\end{pmatrix}
\begin{pmatrix}
1 & l_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & l_1+(l_2-l_1)n \\
0 & 1
\end{pmatrix}
\]

Since the propagation is equivalent to propagation by some extra distance, the far field angle is not changed.
(b) One way to do this is to start at the beam waist, propagate backward by a distance $L_1$, and then forward again through the dielectric interface and then the distance to the new waist. Following example in 6.7, at the original waist we have

$$\frac{1}{g_1} = \frac{1}{kL_1} - i \frac{\lambda}{\pi w_0^2} = -i \frac{\lambda}{\pi w_0^2}$$

From 6.7-6 and $(1 - \Gamma)$ we have

$$g(-L_1) = \frac{g_1 - \Gamma}{1}$$

When we propagate into the material with index $n$, we get

$$g(z) = (g_1 - \Gamma) n$$

$$g(z) = \begin{pmatrix} 0 \\ \frac{1}{n} \end{pmatrix}$$
\[ g_+ = i \frac{\pi \omega_0^2}{\lambda} \ln \frac{\omega_0^2}{\lambda} \]

Propagating forward a distance \( \ell \) to the new waist, we see

\[ g_{\text{new waist}} = i \frac{\pi \omega_0^2}{\lambda} \ln \frac{\omega_0^2}{\lambda} - \ell \ln \frac{\omega_0^2}{\lambda} + 1 \]

so the distance to the new waist is

\[ L = \ell \ln \frac{\omega_0^2}{\lambda} \]

and at that point

\[ g_{\text{new waist}} = i \frac{\pi \omega_0^2}{\lambda} \ln \frac{\omega_0^2}{\lambda} \]

\[ \frac{1}{g_{\text{new waist}}} = -i \frac{\lambda}{\pi \omega_0^2 \ln \frac{\omega_0^2}{\lambda}} \]

So the new waist is the same as the old waist.